# Large amplitude oscillating solutions for three dimensional incompressible Euler equations

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**Abstract.** In this article, we construct large amplitude oscillating waves $(u^\varepsilon)_{\varepsilon\in]0,1]}$ which are *local solutions* on some open domain of the time-space $\mathbb{R}_+\times\mathbb{R}^3$ of both the three dimensional Burger equations (without source term) and the incompressible Euler equations (without pressure). The functions $u^\varepsilon(t,x)$ are mainly characterized by the fact that the corresponding Jacobian matrices $D_x u^\varepsilon(t,x)$ are nilpotent of rank one or two. Our purpose here is to describe the interesting geometrical features of the expressions $u^\varepsilon(t,x)$ obtained by this way.

# 1 Detailed introduction.

## 1.1 Presentation of the framework.

Let $(T,V,r)\in(\mathbb{R}_+^*)^3$ with $T\,V\le r$. We work on a domain of determination having the form of a truncated cone like

$$\Omega_r^T := \big\{(t,x)\in[0,T]\times\mathbb{R}^3\,;\ |x|+t\,V\le r\big\},\qquad |x| := \sqrt{x_1^2+x_2^2+x_3^2}\,.$$

We are looking at expressions $u^\varepsilon(t,x)$, with $\varepsilon\in]0,1]$, which are special solutions of three dimensional Burger equations without source term, namely

$$\partial_t u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon = 0\,,\qquad (t,x)\in\Omega_r^T\subset\mathbb{R}\times\mathbb{R}^3\,. \tag{1.1}$$

We complete (1.1) with a family of oscillating initial data

$$u^\varepsilon(0,x) = h^\varepsilon(x) = \begin{pmatrix} h_1^\varepsilon(x)\\ h_2^\varepsilon(x)\\ h_3^\varepsilon(x)\end{pmatrix} = w\Big(x,\frac{\varphi(x)}{\varepsilon}\Big),\quad (x,\varepsilon)\in\Omega_r^0\times]0,1]\,. \tag{1.2}$$

The function $h^\varepsilon(x)$ is defined on the closed ball $\Omega_r^0$ (having center zero and radius $r$) by using a bounded profile $w(x,\theta)\in\mathcal{C}_b^1(\Omega_r^0\times\mathbb{T};\mathbb{R}^3)$ satisfying

$$\exists\,(x,\theta)\in\Omega_r^0\times\mathbb{T}\,;\qquad \partial_\theta w(x,\theta)\neq 0\,,\qquad \mathbb{T}:=\mathbb{R}/\mathbb{Z}\,. \tag{1.3}$$

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We use also a phase $\varphi \in \mathcal{C}^1(\Omega_r^0;\mathbb{R})$ which is assumed to be not stationary

$$(1.4) \qquad \nabla\varphi(x) := {}^t\big(\partial_1\varphi(x), \partial_2\varphi(x), \partial_3\varphi(x)\big) \neq 0\,, \qquad \forall\, x \in \Omega_r^0\,.$$

The equation (1.1) is the prototype of a quasilinear hyperbolic system. Thus, the solution $u^\varepsilon(t,x)$ of (1.1) which is issued from the bounded initial data $h^\varepsilon(x)$ inherits a finite speed of propagation $V$. In view of (1.2), noting $w = {}^t(w_1, w_2, w_3) \in \mathbb{R}^3$, we can take

$$V \;:=\; sup\, \Big\{ \big( \textstyle\sum_{i=1}^3 w_i(x,\theta)^2 \big)^{1/2}\, ;\; (x,\theta) \in \Omega_r^0 \times \mathbb{T} \Big\} < \infty\,.$$

**Example 1.** *Choose* $T = V = r = 1$. *Select any non constant function* $w_3^e \in \mathcal{C}^\infty(\mathbb{T};\mathbb{R})$ *which is bounded by* 1 *and define*

$$\varphi^e(x) := x_1\,, \qquad w^e(x,\theta) := {}^t\big(0,0,w_3^e(\theta)\big)\,, \qquad u^{e\varepsilon}(x) := {}^t\big(0,0,w_3^e\big(\varphi^e(x)/\varepsilon\big)\big)\,.$$

*Observe that*

$$(1.5) \qquad \partial_t u^{e\varepsilon} + (u^{e\varepsilon}\cdot\nabla)u^{e\varepsilon} \;\equiv\; 0\,, \qquad \operatorname{div} u^{e\varepsilon} \;\equiv\; 0\,, \qquad \big(D_x u^{e\varepsilon}(x)\big)^2 \;\equiv\; 0\,.$$

The expression $u^{e\varepsilon}(x)$ is a very basic example of a contact discontinuity solution of (1.1). More elaborated patterns are proposed in [3, 4, 5, 8, 9, 10]. Extensions can be obtained either by considering nonlinear phases $\varphi$ or by adding some dependence in other variables than $\varphi$. In this article, we explain what can be done in these two directions. More precisely, we construct and classify *all functions* $\varphi(x)$ and $w(x,\theta)$ (if need be, the profile $w$ can also depend in a *smooth way* on $\varepsilon \in [0,1]$) allowing to solve the oscillating Cauchy problem (1.1)-(1.2) in the class of $\mathcal{C}^1-$functions on a domain of determination $\Omega_r^T$ with $(T,r) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ independent of $\varepsilon \in ]0,1]$. Note $D_x h^\varepsilon$ the Jacobian matrix of $h^\varepsilon$, that is

$$D_x h^\varepsilon(x) = \begin{pmatrix} \partial_1 h_1^\varepsilon(x) & \partial_2 h_1^\varepsilon(x) & \partial_3 h_1^\varepsilon(x) \\ \partial_1 h_2^\varepsilon(x) & \partial_2 h_2^\varepsilon(x) & \partial_3 h_2^\varepsilon(x) \\ \partial_1 h_3^\varepsilon(x) & \partial_2 h_3^\varepsilon(x) & \partial_3 h_3^\varepsilon(x) \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}^3)\,.$$

Our starting point is the Theorem 2.6 of [4]. To find on $\Omega_r^T$ a $\mathcal{C}^1-$solution of the Cauchy problem (1.1)-(1.2), it suffices to look at what happens at the initial time $t = 0$. A necessary and sufficient condition is to impose

$$(1.6) \qquad \big(D_x h^\varepsilon(x)\big)^3 = 0\,, \qquad \forall\,(x,\varepsilon) \in \Omega_r^0 \times\, ]0,1]\,.$$

Then (see [4]), the solution of (1.1)-(1.2) satisfies $\operatorname{div} u^\varepsilon = 0$ in $\Omega_r^T$. It means that the solutions of (1.1) under study are also (local) solutions of the incompressible Euler equations (with constant pressure) :

$$(1.7) \qquad \partial_t u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla p_\varepsilon = 0\,, \qquad \operatorname{div} u^\varepsilon = 0\,, \qquad p_\varepsilon = c\,.$$

In what follows, we work with the conditions (1.1), (1.2) and (1.6). We seek *simple wave solutions* meaning that we want to solve *directly* (1.1)-(1.6) through a construction relying on the special form (1.2). Taking into account (1.7), this can be viewed as a preliminary step towards a more general (large amplitude) WKB calculus concerning incompressible or compressible Euler equations. The long-term perspective is indeed to incorporate at the level of (1.1) the influence of extra terms (like pressure, viscosity, $\cdots$) and the presence of complete expansions for the profile such as

$$(1.8) \qquad w_\varepsilon(x,\theta) \,=\, w(x,\theta) + \textstyle\sum_{j=1}^{\infty} \varepsilon^{\kappa\, j} \; w^j(x,\theta)\,, \qquad \kappa \in ]0,1] \cap \mathbb{Q}\,.$$

Let us recall here what has yet been obtained concerning (1.7) when the initial data are adjusted as in (1.2) and (1.8). The case $\kappa = 1$ with a profile $w(x,\theta) = w(x)$ independent of the fast variable $\theta \in \mathbb{T}$ is well-known. It is a variant of standard results in weakly nonlinear geometric optics [11, 13]. The case $\kappa \in ]0,1] \cap \mathbb{Q}$ with still $w(x,\theta) = w(x)$ is fully discussed in [6]. The case $\kappa = 1$ associated with (1.3) corresponds to a more singular situation. It is much more delicate. It is what here holds our attention.

In the case $\kappa = 1$ together with (1.3), the WKB analysis of incompressible Euler equations is supposed to be not well-posed [14]. This is due to a strong coupling between the profile $w(x,\theta)$ and the phase $\varphi(x)$. In such a regime, many unstable phenomena (see for instance [10, 12]) can occur. Therefore, any progress in this direction requires to work in a very specific context, like here (1.1)-(1.2)-(1.6), with adapted tools.

The study of (1.2)-(1.6) is not so easy to achieve. In [3, 4, 7], some very special examples are proposed implying functions $h^\varepsilon(x)$ which are adjusted such that the matrix $D_x h^\varepsilon(x)$ is of rank 1. These preliminary advancements are partially completed in [5] by exploring (without restriction on the space dimension $d \in \mathbb{N}^*_+$) some *necessary* condition on $\varphi(x)$ and $w(x,\theta)$ giving rise to matrices $D_x h^\varepsilon(x)$ which are nilpotent, as in (1.6).

### 1.2 The main results.

In this paper, we restrict our attention to the case $d = 3$ but, this time, we seek *necessary and sufficient* conditions on $(\varphi, w)$ to have (1.2)-(1.6). This approach leads to the notion of *compatible couple* given below.

**Definition 1.1.** *Let $\varphi \in \mathcal{C}^1(\Omega^0_r;\mathbb{R})$ and $w \in \mathcal{C}^1(\Omega^0_r \times \mathbb{T};\mathbb{R}^3)$ two functions satisfying the preliminary assumptions*

$$(1.9) \qquad \partial_\theta w(x,\theta) \,\not\equiv\, 0\,, \qquad \nabla\varphi(x) \,\not\equiv\, 0\,.$$

*The couple $(\varphi, w)$ is said to be* <u>*compatible*</u> *on $\Omega_r^0 \times \mathbb{T}$ if the family $\{h^\varepsilon\}_\varepsilon$ which is associated to $(\varphi, w)$ through (1.2) satisfies (1.6).*

It is possible to derive an exhaustive description of all compatible couples. In the statement below, for the sake of brevity, we express this remarkable fact in a rather imprecise form.

**Theorem 1.** *There is a whole class of compatible couples $(\varphi, w)$.*

The interesting aspects will appear in the text when precising the structure of the functions $\varphi$ and $w$ such involved, and especially when describing the geometrical features of $\varphi$ and how to get them.

Retain here that we can perform a complete WKB analysis of the constraints (1.2), (1.6) and (1.9). Then, applying Theorem 2.6 of [4], we are sure to recover by this way the existence of *large amplitude high-frequency waves* $u^\varepsilon(t, x)$ which are special solutions of (1.7) on $\Omega_r^T$. Now, the structure of the expressions $u^\varepsilon(t, x)$ can be precised as follows.

**Theorem 2.** *Let $(\varphi, w)$ be a couple which is compatible on $\Omega_r^0 \times \mathbb{T}$. There are functions $\mathbf{W}(\varphi, \psi, \theta) \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{T}; \mathbb{R})$ and $\psi(x, \theta) \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R})$ such that the profile $w(x, \theta)$ can be factorized through*

$$w(x, \theta) \,=\, \mathbf{W}\big(\varphi(x), \psi(x, \theta), \theta\big)\,, \qquad \nabla\varphi \wedge \nabla\psi \not\equiv 0\,. \tag{1.10}$$

*There is also some $T > 0$ such that the Cauchy problem*

$$\left\{ \begin{array}{ll} \partial_t \Phi + \big(\mathbf{W}(\Phi, \Psi, \theta) \cdot \nabla\big)\Phi = 0\,, & \Phi(0, x) = \varphi(x)\,, \\ \partial_t \Psi + \big(\mathbf{W}(\Phi, \Psi, \theta) \cdot \nabla\big)\Psi = 0\,, & \Psi(0, x, \theta) = \psi(x, \theta)\,, \end{array} \right. \tag{1.11}$$

*has a solution $(\Phi, \Psi)(t, x, \theta)$ on the domain $\Omega_r^T \times \mathbb{T}$. We have $\partial_\theta \Phi \equiv 0$ and, for all $\varepsilon \in ]0, 1]$, the oscillation*

$$u^\varepsilon(t, x) = \mathbf{W}\big(\Phi(t, x), \Psi(t, x, \Phi(t, x)/\varepsilon), \Phi(t, x)/\varepsilon\big), \quad \varepsilon \in ]0, 1] \tag{1.12}$$

*is a solution of (1.1) on the domain $\Omega_r^T$ with initial data $u^\varepsilon(0, \cdot)$ as in (1.2). Moreover, for all $t \in [0, T]$ the couple $\big(\Phi(t, \cdot), \widetilde{\mathbf{W}}(t, \cdot)\big)$ where*

$$\widetilde{\mathbf{W}}(t, x, \theta) \,:=\, \mathbf{W}\big(\Phi(t, x), \Psi(t, x, \theta), \theta\big)$$

*is still compatible on $B(0, r - tV[\times\mathbb{T}$. More precisely, for all $t \in [0, T]$, we must have*

$$\nabla\Phi \cdot \partial_\theta \mathbf{W} + \partial_\theta \Psi \ \nabla\Phi \cdot \partial_\Psi \mathbf{W} \ \equiv \ 0\,, \tag{1.13}$$

$$(\nabla\Phi \cdot \partial_\Psi \mathbf{W})\ (\nabla\Psi \cdot \partial_\theta \mathbf{W} + \partial_\theta \Psi \ \nabla\Psi \cdot \partial_\Psi \mathbf{W}) \ \equiv \ 0\,, \tag{1.14}$$

$$(\nabla\Phi \cdot \partial_\varphi \mathbf{W})^2 + (\nabla\Phi \cdot \partial_\Psi \mathbf{W})\ (\nabla\Psi \cdot \partial_\varphi \mathbf{W}) \ \equiv \ 0\,, \tag{1.15}$$

$$\nabla\Phi \cdot \partial_\varphi \mathbf{W} + \nabla\Psi \cdot \partial_\Psi \mathbf{W} \ \equiv \ 0\,. \tag{1.16}$$

In comparison with preceding works [3, 4, 5, 7, 10], this second result 2 includes various situations which have not yet been studied. It allows to exhibit many new phenomena with respect to both the propagation and the interaction of oscillations.

### 1.3 Plan of the article.

We present here the plan of the present article. We take this opportunity to make some clarifications and to indicate ideas of proof.

• In Chapter 2, we discuss the notion of *compatible couple*. More precisely, the Proposition 2.1 of Section 2.1 says that any compatible couple $(\varphi, w)$ must verify a list $\mathcal{S}$, namely (2.1)-(2.2)-(2.3)-(2.4), of conditions which are independent of the parameter $\varepsilon \in ]0,1]$.

Then, in the Proposition 2.2 which is proved in Section 2.2, we observe that there exists a scalar function $\psi \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R})$ leading to a factorization of the involved profiles $w(x,\theta)$ in the form (1.10). It follows simplifications when dealing with the system $\mathcal{S}$. It remains (see the Proposition 2.3 proved in Section 2.3) some necessary and sufficient conditions to impose on the three ingredients $\varphi$, $\psi$ and $\mathbf{W}$. In fact, the matter is to work with the relations (1.13), (1.14), (1.15) and (1.16) at the time $t = 0$.

• Chapter 3 consider the simplest case, when $\nabla\varphi \cdot \partial_\psi \mathbf{W} \equiv 0$. Then, as it is explained in Section 3.1, the level surfaces of the phase $\varphi$ can be associated with some foliated structure of $\mathbb{R}^3$ by planes. This information is a crucial key which, in Section 3.2, enables progress leading to a complete description of $(\varphi, \psi, \mathbf{W})$, and therefore $(\varphi, w)$.

• Chapter 4 is devoted to the case $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$. Then, without loss of generality, the profile $w(x,\theta)$ can be assumed to be of the form

$$w(x,\theta) = {}^t\big(v, \psi, \mathfrak{L}(\psi, v)\big)(x,\theta)\,, \qquad v(x,\theta) = \mathbf{V}\big(\varphi(x), \psi(x,\theta), \theta\big)$$

where $\mathfrak{L}(\psi, v)$ and $\mathbf{V}(\varphi, \psi, \theta)$ are auxiliary functions. On the other hand, the expression $\psi(x,\theta)$ can always be factorized according to

$$\psi(x,\theta) \, = \, u\big(x, v(x,\theta)\big)\,, \qquad \partial_v u(x,v) \, \not\equiv \, 0\,.$$

In Section 4.1, see the Proposition 4.1, the information $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$ is exploited in order to rephrase the conditions (1.13), (1.14), (1.15) and (1.16), written at the time $t = 0$ on $\varphi \equiv \Phi(0,\cdot)$, $\psi \equiv \Psi(0,\cdot)$ and $\mathbf{W}$, in terms of the more convenient conditions (4.13), (4.14) and (4.15) which concern only $\varphi$ and $\psi$ (as well as $\mathfrak{L}$ and $\mathbf{V}$).

After eliminating the special case $\partial_3 u \equiv 0$, we concentrate on the remaining situation $\partial_3 u \not\equiv 0$. At this stage, the question becomes the following (see also the remark 4.3.1 for a functional analysis viewpoint).

***The intermediate problem under study.*** *The question is to find smooth* *<u>non</u> <u>constant</u> functions $\Phi(x_1, x_2, u, v)$, locally defined in $\mathbb{R}^4$, satisfying the two transport equations*

$$X\,\Phi \equiv 0\,, \qquad Y\,\Phi \equiv 0\,, \qquad X := \partial_1 + R\,\partial_2\,, \qquad Y := R\,\partial_u + \partial_v \tag{1.17}$$

*and involving a variable coefficient $R(x_1, x_2, u, v)$ which can be identified through the implicit relation*

$$\partial_v u(x, v) \;=\; R\big(x_1, x_2, u(x, v), v\big) \tag{1.18}$$

*where the function $u(x, v)$ must satisfy the two conservation laws*

$$\partial_1 u + \partial_v \mathfrak{L}(u, v)\ \partial_3 u = 0\,, \qquad \partial_2 u + \partial_u \mathfrak{L}(u, v)\ \partial_3 u = 0\,. \tag{1.19}$$

At the level of (1.19), the variable $v$ plays the part of a parameter. When solving (1.19), there are degrees of freedom related to the choices of $u(0, 0, x_3)$ and $\mathfrak{L}(u, v)$. Once the function $u$ (and therefore $R$) is known, the difficulty is to find solutions $\Phi$ of (1.17) satisfying $\nabla\Phi \not\equiv 0$. Let us say a few words about the origin of the conditions (1.17) and $\nabla\Phi \not\equiv 0$. In fact, the expression $\Phi$ is issued from $\varphi$ after a blowing-up procedure. Indeed, one has

$$\varphi(x) \;=\; \Phi\big(x_1, x_2, u(x, v), v\big)\,, \qquad R = \partial_v u \not\equiv 0\,, \qquad v(x, \theta)\,.$$

In this context, the condition $Y\,\Phi \equiv 0$ means simply that $\varphi$ does not depend on $v$. Since the letter $v$ is aimed to be replaced by a function $v(x, \theta)$ of the variables $(x, \theta) \in \mathbb{R}^3 \times \mathbb{T}$, this is equivalent to say that $\partial_\theta \varphi \equiv 0$. This is a natural requirement. Despite the strength of the nonlinearity, we do not want that the phase $\varphi$ starts to oscillate with respect to itself. The other restrictions $X\,\Phi \equiv 0$, (1.18) and (1.19) are coming from (1.6) after the reduction procedure.

Recall that the phase $\varphi$ is supposed to be not stationary, see (1.4). This is possible if and only if the Poisson algebra $\mathcal{A}$ generated by the two vector fields $X$ and $Y$ is of dimension strictly less than four ($\dim \mathcal{A} < 4$). The corresponding integrability criterion (of Frobenius type) can be traducted in terms of conditions on $R$. Actually, the Proposition 4.2 in Section 4.2 exhibits the relevant nonlinear PDE's to impose on $R$. In the case $\dim \mathcal{A} = 2$, we find (4.38). When $\dim \mathcal{A} = 2$, we have to deal with (4.39)-(4.40).

Note that the construction of phases $\varphi$ (through $\Phi$) is associated with the production of special foliations of $\mathbb{R}^4$. The related subtle informations would be out of reach when working with functions $\varphi$ depending only on $x \in \mathbb{R}^3$. Now, the difficulty is that the coefficient $R$ must also be issued from (1.18) after solving the two conservation laws given line (1.19). It follows that the expresssion $R$ inherits some special structure described at the level of Proposition 4.3 in Section 4.3. Given smooth functions $\mathfrak{K}$ and $\mathfrak{L}$, introduce

$$\alpha(x_1, x_2, u, v) := \mathfrak{K}(u, v) + \partial_v \mathfrak{L}(u, v)\ x_1 + \partial_u \mathfrak{L}(u, v)\ x_2 \,. \tag{1.20}$$

The function $R$ must be in the form

$$R(x_1, x_2, u, v) = -\,\partial_v \alpha(x_1, x_2, u, v)/\partial_u \alpha(x_1, x_2, u, v) \,. \tag{1.21}$$

In Section 4.4, we test the integrability conditions (4.39) and (4.40) in the framework of (1.20) and (1.21). Surprisingly, all requirements are met for many choices of the functions $\mathfrak{K}$ and $\mathfrak{L}$ leading in Section 4.5 to a complete classification of all compatible couples $(\varphi, w)$.

To our knowledge, the preceding approach and the corresponding analysis is completely original and new. In the end, it furnishes a good description of the *class* of functions $\varphi(x)$ and $w(x, \theta)$ mentioned in the Theorem 1.

We conclude the chapter 4 by producing in the paragraph 4.6 illustrative examples of compatible couples $(\varphi, w)$.

• In Section 5, we study the time evolution problem. We show Theorem 2. This result is proved in the paragraph 5.1. It furnishes, in the context of the equation (1.1), a complete description of what can happen in terms of smooth large amplitude oscillations. The formula (1.12) generalizes previous examples exhibited in [5, 7, 9, 14].

The families $\{u^\varepsilon\}_{\varepsilon \in ]0,1]}$ exhibited in (1.12) belong to a regime which, in non linear geometric optics, is called supercritical (because one order derivatives of $u^\varepsilon$ explode when $\varepsilon$ goes to 0). Expressions like $u^\varepsilon$ are very unstable objects [7] unless some small viscosity is added [2]. Their asymptotic behaviours (always as $\varepsilon \to 0$) can involve interesting features.

For instance, in Section 5.2, we can exhibit a phenomenon of *superposition* of oscillations. It is obtained by selecting compatible couples $(\varphi, w_\varepsilon)$ where, in contrast to (1.2), the profiles $w_\varepsilon$ depend on $\varepsilon \in ]0, 1]$. More precisely, the expression $w_\varepsilon$ is built with functions $\mathbf{W}$ and $\psi$ through the formula

$$w_\varepsilon(x, \theta) = \mathbf{W}\big(\varphi(x), \psi(x)/\varepsilon, \theta\big) \,, \qquad \mathbf{W}(\varphi, \cdot, \theta) \in \mathcal{C}^\infty(\mathbb{T}; \mathbb{R}^3) \,.$$

At the time $t = 0$, we are faced with a large amplitude multiphase oscillation

$$u^\varepsilon(0,x) = \mathbf{W}\big(\varphi(x), \psi(x)/\varepsilon, \varphi(x)/\varepsilon\big)\,, \qquad \nabla\varphi \wedge \nabla\psi \not\equiv 0\,. \tag{1.22}$$

On the other hand, at any time $t \in ]0,T]$, the function $\Psi(t,\cdot)$ starts to really depend on $\theta \in \mathbb{T}$ giving rise to

$$u^\varepsilon(t,x) = \mathbf{W}\Big(\Phi(t,x), \frac{\Psi\big(t,x,\Phi(t,x)/\varepsilon\big)}{\varepsilon}, \frac{\Phi(t,x)}{\varepsilon}\Big)\,. \tag{1.23}$$

Thus, the interaction of large amplitude waves oscillating in transversal directions at the frequency $\varepsilon^{-1}$ can produce oscillations with frequency $\varepsilon^{-2}$. Such a *turbulent* effect was already mentioned in [3] in the context of the system (1.1) when $d = 2$. On the contrary, we were not able to prove the same effect in the case of (1.7) when $d = 2$. It seems that, when adding the divergence free condition, it is a specificity of the space dimension $d = 3$.

• The aim of Appendix 6 is to check that the list of situations enumerated at the level of Proposition 4.4 is exhaustive. The corresponding work of verification is quite long and technical. The difficulties are due to the fact that it is delicate to interpret the integrability conditions to impose on $R$ into convenient constraints on the functions $\mathfrak{K}$ and $\mathfrak{L}$ appearing at the level of (1.20). This will be done step by step, from paragraph 6.1 up to 6.5.

# Contents

## 2 Compatible couples.

From now on, we write $f \equiv 0$ and $f \not\equiv 0$ to mean respecively that $f$ is identically zero on its domain of definition or that it is a non-zero function.

## 2.1 The notion of compatible couples.

Given two vectors $u = {}^t(u_1, u_2, u_3) \in \mathbb{R}^3$ and $v = {}^t(v_1, v_2, v_3) \in \mathbb{R}^3$, we note

$$u \cdot v \;:=\; u_1\, v_1 + u_2\, v_2 + u_3\, v_3 \,,$$

$$u \otimes v := \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}, \qquad u \wedge v := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

We can interpret (1.6) in the form of conditions on $(w, \varphi)$.

**Proposition 2.1.** *Let $\varphi \in C^1(\Omega_r^0; \mathbb{R})$ and $w \in C^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R}^3)$ satisfying the preliminary assumptions (1.9). The couple $(\varphi, w)$ is compatible on $\Omega_r^0 \times \mathbb{T}$ if and ony if it is a solution on $\Omega_r^0 \times \mathbb{T}$ of the system $\mathcal{S}$ made of*

$$\nabla\varphi \cdot \partial_\theta w \;\equiv\; 0 \,, \tag{2.1}$$

$$\nabla\varphi \cdot (D_x w \; \partial_\theta w) \;\equiv\; 0 \,, \tag{2.2}$$

$$(D_x w)^3 \;\equiv\; 0 \,, \tag{2.3}$$

$$M\,(D_x w)^2 + D_x w\, M\, D_x w + (D_x w)^2\, M \;\equiv\; 0 \,, \quad M := \partial_\theta w \otimes \nabla\varphi \,. \tag{2.4}$$

***Proof of Proposition 2.1.*** We find

$$D_x h^\varepsilon(x) = (D_x w)\Big(x, \frac{\varphi(x)}{\varepsilon}\Big) + \frac{1}{\varepsilon}\, \partial_\theta w\Big(x, \frac{\varphi(x)}{\varepsilon}\Big) \otimes \nabla\varphi(x) \,.$$

The constraint (1.6) can also be formulated as

$$\sum_{j=0}^{3} \varepsilon^{-j}\, \Xi_j\Big(x, \frac{\varphi(x)}{\varepsilon}\Big) \equiv 0 \,, \qquad \Xi_j(x,\theta) \in C^0\big(\Omega_r^0 \times \mathbb{T}, \mathcal{M}_3(\mathbb{R}^3)\big)$$

where

$$\begin{array}{ll} \Xi_0 = (D_x w)^3 \;, & \Xi_1 = (D_x w)^2\, M + D_x w\, M\, D_x w + M\,(D_x w)^2 \,, \\ \Xi_3 = M^3 \qquad , & \Xi_2 = M^2\, D_x w + D_x w\, M^2 + M\, D_x w\, M \,. \end{array}$$

To guarantee (1.6) for all $\varepsilon \in ]0,1]$, it is necessary and sufficient to impose

$$\Xi_j \equiv 0 \,, \qquad \forall\, (x,\theta) \in \Omega_r^0 \times \mathbb{T} \,, \qquad \forall\, j \in \{0,1,2,3\} \;. \tag{2.5}$$

Our aim is to solve (2.5) for some $r \in \mathbb{R}_+^*$. The constraints $\Xi_0 \equiv 0$ and $\Xi_1 \equiv 0$ are repetitions of respectively (2.3) and (2.4). Since

$$M^3 = (\nabla\varphi \cdot \partial_\theta w)^2\; \partial_\theta w \otimes \nabla\varphi \;=\; 0 \,, \qquad \partial_\theta w \otimes \nabla\varphi \not\equiv 0 \,,$$

the examination of $\Xi_3$ leads to (2.1). In view of (2.1), we have also $M^2 \equiv 0$. Thus, the condition $\Xi_2 \equiv 0$ reduces to $M\, D_x w\, M \equiv 0$, that is (2.2). $\square$

The system $\mathcal{S}$, as presented above, is not yet exploitable. The purpose of this chapter 2 is to put it in a suitable form. In view of (2.3), the rank of the matrix $D_x w$ is either one or two (the zero case being trivial). The next paragraphs 2.1.1 and 2.1.2 deal separately with these two situations.

### 2.1.1 The case of rank one.

In this paragraph, we suppose that

$$rg\big(D_x w(x,\theta)\big) = dim\big(Im(D_x w)(x,\theta)\big) = 1\,, \qquad \forall\,(x,\theta) \in \Omega_r^0 \times \mathbb{T}\,. \tag{2.6}$$

By the constant rank theorem [1] and due to the compacity of the torus $\mathbb{T}$, by restricting $r \in \mathbb{R}_+^*$ if necessary, we can find two functions $\psi \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R})$ and $\mathbf{W} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{T}; \mathbb{R}^3)$ with $\nabla\psi \not\equiv 0$ and $\partial_\psi \mathbf{W} \not\equiv 0$ such that

$$w(x,\theta) = \mathbf{W}\big(\psi(x,\theta),\theta\big), \qquad \forall\,(x,\theta) \in \Omega_r^0 \times \mathbb{T}\,. \tag{2.7}$$

**Lemma 2.1.** *Assume (1.9) and (2.7). Then, the couple $(\varphi, w)$ is compatible on the domain $\Omega_r^0 \times \mathbb{T}$ if and only if the following conditions are verified :*

$$\nabla\varphi \cdot \partial_\theta w \;\equiv\; 0\,, \tag{2.8}$$

$$(\nabla\varphi \cdot \partial_\psi \mathbf{W})\,(\nabla\psi \cdot \partial_\theta w) \;\equiv\; 0\,, \tag{2.9}$$

$$\nabla\psi \cdot \partial_\psi \mathbf{W} \;\equiv\; 0\,. \tag{2.10}$$

***Proof of Lemma 2.1.*** The condition (2.8) is the same as (2.1). Taking into account (2.7), we find $D_x w = \partial_\psi \mathbf{W} \otimes \nabla\psi$ so that (2.3) becomes

$$(\nabla\psi \cdot \partial_\psi \mathbf{W})^2\; \partial_\psi \mathbf{W} \otimes \nabla\psi \equiv 0\,, \qquad \partial_\psi \mathbf{W} \otimes \nabla\psi \not\equiv 0$$

which implies (2.10). Knowing (2.10), the constraint (2.4) reduces to

$$D_x w\, M\, D_x w = (\nabla\varphi \cdot \partial_\psi \mathbf{W})\,(\nabla\psi \cdot \partial_\theta w)\; \partial_\psi \mathbf{W} \otimes \nabla\psi \equiv 0\,.$$

We recover here (2.9) which also allows to guarantee (2.2). $\square$

### 2.1.2 The case of rank two.

In this paragraph, we suppose that

$$rg\big(D_x w(x,\theta)\big) = dim\big(Im(D_x w)(x,\theta)\big) = 2, \qquad \forall\,(x,\theta) \in \Omega_r^0 \times \mathbb{T}\,. \tag{2.11}$$

As before, we can apply the constant rank theorem [1] in order to find three functions $\psi \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R})$, $\tilde{\psi} \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R})$ and $\mathbf{W} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{T}; \mathbb{R}^3)$ with $\nabla\psi \not\equiv 0$, $\nabla\tilde{\psi} \not\equiv 0$, $\nabla\psi \wedge \nabla\tilde{\psi} \not\equiv 0$ and $\partial_\psi \mathbf{W} \not\equiv 0$ such that

$$w(x,\theta) = \mathbf{W}\big(\tilde{\psi}(x,\theta), \psi(x,\theta), \theta\big), \qquad \forall\,(x,\theta) \in \Omega_r^0 \times \mathbb{T}\,. \tag{2.12}$$

In the Section 2.2, we will show that we can take $\tilde{\psi} \equiv \varphi$. The precise statement is the following.

**Proposition 2.2.** *Let $(\varphi, w)$ be a compatible couple on the domain $\Omega_r^0 \times \mathbb{T}$. By restricting $r \in \mathbb{R}_+^*$ if necessary, we can find a function $\psi \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}; \mathbb{R})$ satisfying $\nabla\varphi \wedge \nabla\psi \not\equiv 0$ and a vector function $\mathbf{W} \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{T}; \mathbb{R}^3)$ such as*

$$w(x,\theta) = \mathbf{W}\big(\varphi(x), \psi(x,\theta), \theta\big)\,, \qquad \forall\ (x,\theta) \in \Omega_r^0 \times \mathbb{T}\,. \tag{2.13}$$

Assuming (2.13), we can compute

$$D_x w(x,\theta) = \partial_\varphi \mathbf{W} \otimes \nabla\varphi + \partial_\psi \mathbf{W} \otimes \nabla\psi\,. \tag{2.14}$$

In view of (2.11), the two vectors $\nabla\varphi$ and $\nabla\psi$, as well as $\partial_\varphi \mathbf{W}$ and $\partial_\psi \mathbf{W}$, must be independent. In other words :

$$\nabla\varphi \wedge \nabla\psi \not\equiv 0\,, \qquad \partial_\varphi \mathbf{W} \wedge \partial_\psi \mathbf{W} \not\equiv 0\,. \tag{2.15}$$

On the other hand, the condition (1.9) amounts to the same thing as

$$\partial_\theta \psi\ \partial_\psi \mathbf{W} + \partial_\theta \mathbf{W} \not\equiv 0\,. \tag{2.16}$$

In the Section 2.3, we will further exploit the information (2.13) in order to interpret the system $\mathcal{S}$ differently. Just retain here that :

**Proposition 2.3.** *Assume (2.11) and (2.13) together with the preliminary hypothesis (2.16). Then, the couple $(\varphi, w)$ is compatible on the domain $\Omega_r^0 \times \mathbb{T}$ if and only if we have (2.15) and the following conditions :*

$$\nabla\varphi \cdot \partial_\theta \mathbf{W} + \partial_\theta \psi\ \nabla\varphi \cdot \partial_\psi \mathbf{W} \equiv 0\,, \tag{2.17}$$

$$(\nabla\varphi \cdot \partial_\psi \mathbf{W})\ (\nabla\psi \cdot \partial_\theta \mathbf{W} + \partial_\theta \psi\ \nabla\psi \cdot \partial_\psi \mathbf{W}) \equiv 0\,, \tag{2.18}$$

$$(\nabla\varphi \cdot \partial_\varphi \mathbf{W})^2 + (\nabla\varphi \cdot \partial_\psi \mathbf{W})\ (\nabla\psi \cdot \partial_\varphi \mathbf{W}) \equiv 0\,, \tag{2.19}$$

$$\nabla\varphi \cdot \partial_\varphi \mathbf{W} + \nabla\psi \cdot \partial_\psi \mathbf{W} \equiv 0\,. \tag{2.20}$$

Comparing the two Propositions 2.1 and 2.3, we see that (2.8)-(2.9)-(2.10) can be handled as a special case of (2.17)-(2.18)-(2.19)-(2.20). It suffices to work with $\partial_\varphi \mathbf{W} \equiv 0$. Thus, in the chapters 3 and 4, we can concentrate on the system (2.17)-(2.18)-(2.19)-(2.20). We will examine separately what happens when respectively $\nabla\varphi \cdot \partial_\psi \mathbf{W} \equiv 0$ and $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$.

## 2.2 Factorization of compatible couples.

Suppose (2.11). To obtain (2.13), we proceed in two steps. First, in the paragraph 2.2.1, we produce a local version of the Proposition 2.2. Then, in the paragraph 2.2.2, we complete the proof of the Proposition 2.2.

### 2.2.1 The local version of the Proposition 2.2.

Note $\vec{0} = (0,0,0) \in \Omega_r^0 \subset \mathbb{R}^3$. In this paragraph, we work locally, near a point $(\vec{0},\tilde{\theta}) \in \Omega_r^0 \times \mathbb{T}$. We select some open connected neighbourhood $\Gamma$ satisfying $(\vec{0},\tilde{\theta}) \in \Gamma \subset \Omega_r^0 \times \mathbb{T}$. Typically, we can take

$$\Gamma \equiv \Gamma_{r,\tilde{r}}^{\tilde{\theta}} := \Omega_r^0 \times ]\tilde{\theta} - \tilde{r}\,, \tilde{\theta} + \tilde{r}[\,, \qquad (r,\tilde{r},\tilde{\theta}) \in \mathbb{R}_+^* \times ]0,1[\, \times \mathbb{T}\,.$$

Let $(\varphi, w)$ be a couple which is compatible on $\Gamma_{r,\tilde{r}}^{\tilde{\theta}}$. By exchanging $w(x,\theta)$ into $w(x,\theta-\tilde{\theta})$, we can always suppose that $\tilde{\theta} = 0$. In what follows, we will argue on $\Gamma_{r,\tilde{r}}^0$. Note $i$, $j$ and $k$ three distinct elements chosen among the set $\{1,2,3\}$. The constraint (2.11) means that there is $k$ giving rise to

$$\nabla w_k(x,\theta) \in Vec\,\langle \nabla w_i(x,\theta), \nabla w_j(x,\theta)\rangle \quad , \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0\,, \tag{2.21}$$

$$\nabla w_i(x,\theta) \wedge \nabla w_j(x,\theta) \neq 0 \quad , \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0\,. \tag{2.22}$$

The direction $\nabla \varphi(\vec{0})$ cannot be simultaneously colinear to the two vectors $\nabla w_i(\vec{0},0)$ and $\nabla w_j(\vec{0},0)$. Pick the indice $l \in \{i,j\}$ in such a way that $\nabla\varphi(\vec{0}) \wedge \nabla w_l(\vec{0},0) \neq 0$. Then, do a permutation on the three directions $x_1$, $x_2$ and $x_3$ (with the corresponding permutation on the components $w_1$, $w_2$ and $w_3$) in order to have $l=1$ and $k=3$. Then, by restricting $r \in \mathbb{R}_+^*$ and $\tilde{r} \in ]0,1[$, we can obtain

$$\nabla\varphi \wedge \nabla w_1 \not\equiv 0\,, \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0 \tag{2.23}$$

while the conditions (2.21) and (2.22) become

$$\nabla w_3(x,\theta) \in Vec\,\langle \nabla w_1(x,\theta), \nabla w_2(x,\theta)\rangle \quad , \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0\,, \tag{2.24}$$

$$\nabla w_1(x,\theta) \wedge \nabla w_2(x,\theta) \neq 0 \quad , \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0\,. \tag{2.25}$$

The constraint (2.24) allows to deduce the existence of a scalar function $\mathbb{W}_3$ in $C^1(\mathbb{R}^2 \times ]-\tilde{r},\tilde{r}[;\mathbb{R})$ such that

$$w_3(x,\theta) = \mathbb{W}_3\big(w_1(x,\theta), w_2(x,\theta), \theta\big), \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0\,. \tag{2.26}$$

Then, using the convention

$$\mathbb{W}(w_1,w_2,\theta) = \begin{pmatrix} \mathbb{W}_1(w_1,w_2,\theta) \\ \mathbb{W}_2(w_1,w_2,\theta) \\ \mathbb{W}_3(w_1,w_2,\theta) \end{pmatrix} := \begin{pmatrix} w_1 \\ w_2 \\ \mathbb{W}_3(w_1,w_2,\theta) \end{pmatrix},$$

we can get

$$w(x,\theta) = \mathbb{W}\big(w_1(x,\theta), w_2(x,\theta), \theta\big), \quad \forall\,(x,\theta) \in \Gamma_{r,\tilde{r}}^0\,. \tag{2.27}$$

**Lemma 2.2.** *Select a couple* $(\varphi, w)$ *which is compatible on* $\Gamma^0_{r,\tilde r}$ *and which satisfies (2.24) together with (2.25). Then, there exists a scalar function* $W_2 \in C^1(\mathbb{R}^2 \times ]-\tilde r, \tilde r[;\mathbb{R})$ *such that the component* $w_2$ *can be put in the form*

$$w_2(x,\theta) = W_2\big(\varphi(x), w_1(x,\theta), \theta\big), \quad \forall\, (x,\theta) \in \Gamma^0_{r,\tilde r}\,. \tag{2.28}$$

***Proof of Lemma 2.2.*** To obtain (2.28), it suffices to show that

$$\nabla w_2(x,\theta) \in Vec\,\langle \nabla\varphi(x), \nabla w_1(x,\theta)\rangle\,, \quad \forall\, (x,\theta) \in \Gamma^0_{r,\tilde r}\,. \tag{2.29}$$

The proof is by contradiction. Suppose that (2.29) is not verified :

$$\exists\, (x_0,\theta_0) \in \Gamma^0_{r,\tilde r}\,, \qquad \nabla w_2(x_0,\theta_0) \notin Vec\,\langle \nabla\varphi(x_0), \nabla w_1(x_0,\theta_0)\rangle\,. \tag{2.30}$$

Combining (2.23) and (2.30), we see that the vectors $\nabla\varphi(x_0)$, $\nabla w_1(x_0,\theta_0)$ and $\nabla w_2(x_0,\theta_0)$ give rise to a basis of $\mathbb{R}^3$. In addition, by using the definition of the $\Xi_j$ and the restrictions (2.1), (2.2), (2.3) and (2.4), we can get

$$\big(D_x w + \partial_\theta w \otimes \nabla\varphi\big)^3 = \textstyle\sum_{j=0}^3 \; \Xi_j = 0\,.$$

Thus, the matrice

$$D_x w + \partial_\theta w \otimes \nabla\varphi = \begin{pmatrix} {}^t\nabla w_1 + \partial_\theta w_1 \; {}^t\nabla\varphi \\ {}^t\nabla w_2 + \partial_\theta w_2 \; {}^t\nabla\varphi \\ {}^t\nabla w_3 + \partial_\theta w_3 \; {}^t\nabla\varphi \end{pmatrix}$$

is at most of rank two. In view of (2.26), the third row vector is

$$\begin{aligned} \nabla w_3 + \partial_\theta w_3 \;\nabla\varphi \quad &= \;\partial_{w_1}\mathbb{W}_3 \;(\nabla w_1 + \partial_\theta w_1 \;\nabla\varphi) \\ &\quad + \partial_{w_2}\mathbb{W}_3 \;(\nabla w_2 + \partial_\theta w_2 \;\nabla\varphi) + \partial_\theta\mathbb{W}_3\,\nabla\varphi\,. \end{aligned}$$

It must be a linear combination of the two first row vectors so that

$$(\partial_\theta \mathbb{W}_3)\big(w_1(x_0,\theta_0), w_2(x_0,\theta_0), \theta_0\big) = 0. \tag{2.31}$$

In what follows, the functions will be (unless stated otherwise) computed at the point $(x,\theta) = (x_0,\theta_0)$. The information (2.31) implies that

$$\partial_\theta w_3 \;=\; \partial_{w_1}\mathbb{W}_3(w_1,w_2,\theta_0)\;\partial_\theta w_1 \;+\; \partial_{w_2}\mathbb{W}_3(w_1,w_2,\theta_0)\;\partial_\theta w_2\,.$$

Looking at (1.9), we note that either $\partial_\theta w_1(x_0,\theta_0) \neq 0$ or $\partial_\theta w_2(x_0,\theta_0) \neq 0$. We will below consider the case $\partial_\theta w_2(x_0,\theta_0) \neq 0$. The other situation (that is $\partial_\theta w_1 \neq 0$) can be dealt in a similar way.

The constraint (2.31) allows simplifications when writing (2.1), (2.2), (2.3) and (2.4). For example, the condition (2.1) reduces to

$$\nabla\varphi \cdot \partial_{w_2}\mathbb{W} = -\,\frac{\partial_\theta w_1}{\partial_\theta w_2}\;\nabla\varphi \cdot \partial_{w_1}\mathbb{W}. \tag{2.32}$$

The condition (2.3) is nothing other than

$$(D_x w)^3 = \big[(D_x w)^2\, \partial_{w_1} \mathbb{W}\big] \otimes \nabla w_1 + \big[(D_x w)^2\, \partial_{w_2} \mathbb{W}\big] \otimes \nabla w_2 \,\equiv\, 0\,.$$

Taking into account (2.25), this identity is possible only if

$$(D_x w)^2\, \partial_{w_1} \mathbb{W} \,\equiv\, 0, \qquad (D_x w)^2\, \partial_{w_2} \mathbb{W} \,\equiv\, 0. \tag{2.33}$$

Defining $\alpha := {}^t\nabla w_1\, D_x w\, \partial_{w_1}\mathbb{W}$ and $\beta := {}^t\nabla w_2\, D_x w\, \partial_{w_1}\mathbb{W}$, we find

$$(D_x w)^2\, \partial_{w_1} \mathbb{W} \,=\, \alpha\; {}^t(1, 0, \partial_{w_1}\mathbb{W}_3) \,+\, \beta\; {}^t(0, 1, \partial_{w_2}\mathbb{W}_3)\,.$$

The first constraint of (2.33) means that the two coefficients $\alpha$ and $\beta$ are zero, yielding

$$(\nabla w_1 \cdot \partial_{w_1}\mathbb{W})^2 + (\nabla w_1 \cdot \partial_{w_2}\mathbb{W})\,(\nabla w_2 \cdot \partial_{w_1}\mathbb{W}) \;=\; 0\,, \tag{2.34}$$

$$(\nabla w_2 \cdot \partial_{w_1}\mathbb{W})\,(\nabla w_1 \cdot \partial_{w_1}\mathbb{W} + \nabla w_2 \cdot \partial_{w_2}\mathbb{W}) \;=\; 0\,. \tag{2.35}$$

By the same method followed this time at the level of the second condition, we can extract the necessary and sufficient conditions

$$(\nabla w_1 \cdot \partial_{w_2}\mathbb{W})\,(\nabla w_1 \cdot \partial_{w_1}\mathbb{W} + \nabla w_2 \cdot \partial_{w_2}\mathbb{W}) \;=\; 0\,, \tag{2.36}$$

$$(\nabla w_2 \cdot \partial_{w_2}\mathbb{W})^2 + (\nabla w_1 \cdot \partial_{w_2}\mathbb{W})\,(\nabla w_2 \cdot \partial_{w_1}\mathbb{W}) \;=\; 0\,. \tag{2.37}$$

We claim that it is not possible to have

$$\nabla w_1 \cdot \partial_{w_1}\mathbb{W} + \nabla w_2 \cdot \partial_{w_2}\mathbb{W} \neq 0. \tag{2.38}$$

Indeed, suppose that (2.38) is true. Then, the relations (2.35) and (2.36) imply that $\nabla w_2 \cdot \partial_{w_1}\mathbb{W} = 0$ and that $\nabla w_1 \cdot \partial_{w_2}\mathbb{W} = 0$. Using these informations, the relations (2.34) and (2.37) lead to $\nabla w_1 \cdot \partial_{w_1}\mathbb{W} = 0$ and $\nabla w_2 \cdot \partial_{w_2}\mathbb{W} = 0$. Now, these two last informations are in contradiction with (2.38). Therefore, we are sure that

$$\nabla w_1 \cdot \partial_{w_1}\mathbb{W} + \nabla w_2 \cdot \partial_{w_2}\mathbb{W} = 0. \tag{2.39}$$

The condition (2.39) induces (2.35) and (2.36). It is also adjusted in such a way that (2.37) is equivalent to (2.34). Thus, the analysis of (2.3) is the same as the one of (2.34) and (2.39). These two constraints (2.34) and (2.39) say in particular that the two vectors

$$(\nabla w_1 \cdot \partial_{w_1}\mathbb{W}, \nabla w_2 \cdot \partial_{w_1}\mathbb{W}) \in \mathbb{R}^2, \qquad (\nabla w_1 \cdot \partial_{w_2}\mathbb{W}, \nabla w_2 \cdot \partial_{w_2}\mathbb{W}) \in \mathbb{R}^2$$

are colinear. In other words, we can find $(\tilde\alpha,\ \tilde\beta) \in \mathbb{R}^2\backslash(0,0)$ such that

$$\nabla w_1 \cdot (\tilde\alpha\, \partial_{w_1}\mathbb{W} + \tilde\beta\, \partial_{w_2}\mathbb{W}) = 0, \qquad \nabla w_2 \cdot (\tilde\alpha\, \partial_{w_1}\mathbb{W} + \tilde\beta\, \partial_{w_2}\mathbb{W}) = 0. \tag{2.40}$$

Now, we consider (2.2) computed at $(x_0, \theta_0)$. Exploiting the informations (2.31), (2.32) and (2.39), we can formulate (2.2) according to

$$
\begin{aligned}
\Big[ 2\, \partial_\theta w_1 \, (\nabla w_1 \cdot \partial_{w_1} \mathbb{W}) + \partial_\theta w_2 \, (\nabla w_1 \cdot \partial_{w_2} \mathbb{W}) & \\
- \frac{(\partial_\theta w_1)^2}{\partial_\theta w_2} \, (\nabla w_2 \cdot \partial_{w_1} \mathbb{W}) \Big] \, (\nabla \varphi \cdot \partial_{w_1} \mathbb{W}) &= 0.
\end{aligned} \tag{2.41}
$$

Multiply (2.41) by $\partial_\theta w_2 \, (\nabla w_2 \cdot \partial_{w_1} \mathbb{W})$. Then, use (2.34) and (2.39) to obtain

$$
(\nabla \varphi \cdot \partial_{w_1} \mathbb{W}) \, (\nabla w_2 \cdot \partial_\theta w)^2 = 0 \, . \tag{2.42}
$$

In the same way, multiply (2.41) by $\partial_\theta w_2 \, (\nabla w_1 \cdot \partial_{w_2} \mathbb{W})$. Then, use (2.34) in order to extract

$$
(\nabla \varphi \cdot \partial_{w_1} \mathbb{W}) \, (\nabla w_1 \cdot \partial_\theta w)^2 = 0 \, . \tag{2.43}
$$

Two situations can happen :

$\triangleright 1 \triangleleft$ <u>*The case*</u> $\nabla \varphi \cdot \partial_{w_1} \mathbb{W} \neq 0$. The equations (2.1), (2.42) and (2.43) imply that $\nabla \varphi$, $\nabla w_1$ and $\nabla w_2$ belong to the same plane $(\partial_\theta w^\perp)$. It follows that these vectors are linearly dependent, in contradiction with (2.30).

$\triangleright 2 \triangleleft$ <u>*The case*</u> $\nabla \varphi \cdot \partial_{w_1} \mathbb{W} = 0$. From (2.32), we deduce that $\nabla \varphi \cdot \partial_{w_2} \mathbb{W} = 0$. It follows that

$$
\nabla \varphi \cdot (\alpha' \, \partial_{w_1} \mathbb{W} + \beta' \, \partial_{w_2} \mathbb{W}) = 0, \quad \forall \, (\alpha', \beta') \in \mathbb{R}^2. \tag{2.44}
$$

We choose $\alpha' = \tilde{\alpha}$ and $\beta' = \tilde{\beta}$. According to the definition of the function $\mathbb{W}$ and since $(\tilde{\alpha}, \tilde{\beta}) \neq (0,0)$, we have

$\tilde{\alpha} \, \partial_{w_1} \mathbb{W} + \tilde{\beta} \, \partial_{w_2} \mathbb{W} = {}^t(\tilde{\alpha}, \tilde{\beta}, \star) \neq (0,0,0) \, .$

The informations (2.40) and (2.44) (where $\alpha' = \tilde{\alpha}$ and $\beta' = \tilde{\beta}$) indicate that the vectors $\nabla \varphi$, $\nabla w_1$ and $\nabla w_2$ belong to the same plane, namely $(\tilde{\alpha} \, \partial_{w_1} \mathbb{W} + \tilde{\beta} \, \partial_{w_2} \mathbb{W})^\perp$. It follows that these three vectors are linearly dependent. This is in contradiction with (2.30).

In conclusion, we have (2.29), as expected. $\square$

**Proposition 2.4** (local version of the Proposition 2.2)**.** *Assume (2.11) and select any* $\tilde{\theta} \in \mathbb{T}$. *Let* $(\varphi, w)$ *be a couple which is compatible on* $\Gamma^{\tilde{\theta}}_{r,\tilde{r}}$. *Then, by selecting* $r \in \mathbb{R}^*_+$ *and* $\tilde{r} \in ]0,1[$ *conveniently and by permuting the directions*

*$x_1, x_2$ and $x_3$ (with accordingly the components $w_1,\ w_2$ and $w_3$ of $w$), it is possible to obtain (2.23) and to write the profile $w(x,\theta)$ in the form*

$$w(x,\theta) = W\big(\varphi(x), w_1(x,\theta), \theta\big)\,, \qquad (x,\theta) \in \Gamma_{r,\tilde r}^{\tilde\theta} \tag{2.45}$$

*with a function $W = {}^t(W_1, W_2, W_3) \in C^1(\mathbb{R}^2 \times ]-\tilde r, \tilde r[; \mathbb{R}^3)$ whose two first components $W_1$ and $W_2$ satisfy*

$$W_1(\varphi, w_1, \theta) \,=\, w_1\,, \qquad \forall\,(\varphi, w_1,\theta) \in \mathbb{R}^2 \times ]-\tilde r, \tilde r[\,, \tag{2.46}$$

$$\partial_\varphi W_2(\varphi, w_1, \theta) \,\neq\, 0\,, \qquad \forall\,(\varphi, w_1,\theta) \in \mathbb{R}^2 \times ]-\tilde r, \tilde r[\,. \tag{2.47}$$

***Proof of proposition 2.4.*** Without loss of generality, we can suppose that $\tilde\theta = 0$. Taking into account (2.28), the function $w_3$ can be put in the form

$$w_3(x,\theta) = W_3\big(\varphi(x), w_1(x,\theta), \theta\big), \qquad \forall\,(x,\theta) \in \Gamma_{r,\tilde r}^0$$

with $W_3(\varphi, w_1, \theta) := \mathbb{W}_3\big(w_1, W_2(\varphi, w_1,\theta), \theta\big)$. In addition, we can define

$$W_1(\varphi, w_1, \theta) := w_1\,, \qquad \forall\,(\varphi, w_1,\theta) \in \mathbb{R}^2 \times ]-\tilde r, \tilde r[\,.$$

With these conventions, we recover both (2.45) and (2.46). Recalling (2.23), to have (2.11), the vector $\partial_\varphi W \wedge \partial_{w_1} W$ must not vanish on $\mathbb{R}^2 \times ]-\tilde r, \tilde r[$. This amounts to saying that the function $\partial_\varphi W_2$ does not vanish on $\mathbb{R}^2 \times ]-\tilde r, \tilde r[$. This is exactly what requires the condition (2.47). $\square$

### 2.2.2 The proof of the Proposition 2.2.

Select a compatible couple $(\varphi, w)$. The condition (2.11) implies that

$$dim\, Vec\,\langle \nabla w_1, \nabla w_2, \nabla w_3\rangle = 2, \qquad \forall\,(x,\theta) \,\in\, \Omega_r^0 \times \mathbb{T}\,. \tag{2.48}$$

Locally, by permuting the directions $x_1$, $x_2$ and $x_3$ as it is made in the Proposition 2.4, we can get $\nabla\varphi \in Vec\,\langle \nabla w_1, \nabla w_2\rangle$. It follows that the direction $\nabla\varphi$ belongs to the vector space $Vec\,\langle \nabla w_1, \nabla w_2, \nabla w_3\rangle$. Observe that this property does not depend on the choice of the coordinates. Thus, it remains to be true in all the domain under study. We must have

$$\nabla\varphi \,\in\, Vec\,\langle \nabla w_1, \nabla w_2, \nabla w_3\rangle\,, \qquad \forall\,(x,\theta) \,\in\, \Omega_r^0 \times \mathbb{T}\,. \tag{2.49}$$

Fix $\theta \in \mathbb{T}$. Given a function $\Psi_\theta \in C^1(\mathbb{R}^3;\mathbb{R})$ and $r_\theta \in ]0, r[$, introduce

$$\psi_\theta(x,\tilde\theta) \,:=\, \Psi_\theta(w_1, w_2, w_3)(x,\tilde\theta)\,, \qquad \forall\,(x,\tilde\theta) \in \Omega_{r_\theta}^0 \times ]\theta - r_\theta, \theta + r_\theta[\,.$$

We can deduce from (2.48) and (2.49) the existence of $\Psi_\theta \in C^1(\mathbb{R}^3;\mathbb{R})$ and $r_\theta \in ]0, r[$ such that $\nabla\varphi$ is not colinear to $\nabla\psi_\theta$, namely that the first component of $\nabla\varphi \wedge \nabla\psi_\theta$ is positive

$$(\nabla\varphi \wedge \nabla\psi_\theta)_1 \, > \, 0 \, , \qquad \forall\, (x,\theta) \in \Omega^0_{r_\theta} \times ]\theta - r_\theta, \theta + r_\theta[$$

whereas

$$Vec\,\langle \nabla\varphi, \nabla\psi_\theta\rangle \equiv Vec\,\langle \nabla w_1, \nabla w_2, \nabla w_3\rangle \, , \quad \forall\, (x,\theta) \in \Omega^0_{r_\theta} \times ]\theta - r_\theta, \theta + r_\theta[\, .$$

The family of intervals $]\theta - r_\theta, \theta + r_\theta[$ with $\theta \in \mathbb{T}$ is an open cover of $\mathbb{T}$. Since $\mathbb{T}$ is compact, there is a finite subcover $\mathbb{T} \subset \bigcup_{i=1}^N ]\theta_i - r_{\theta_i}, \theta_i + r_{\theta_i}[\,$. Now, consider some associated partition of unity $\{\chi_i\}_{i=1}^N$ where the functions $\chi_i \in C^\infty(\mathbb{T};\mathbb{R}_+)$ are adjusted such that $supp\,\chi_i \subset ]\theta_i - r_{\theta_i}, \theta_i + r_{\theta_i}[$ and $\sum_{i=1}^N \chi_i \equiv 1$. We replace $r \in \mathbb{R}_+^*$ by the minimum of the numbers $r_{\theta_i}$ (with $i \in \{1, \cdots, N\}$). Then, we can introduce $\psi(x,\theta) := \sum_{i=1}^N \psi_{\theta_i}(x,\theta)\,\chi_i(\theta)$. The preceding construction yields (2.15) as well as

$$Vec\,\langle \nabla\varphi, \nabla\psi\rangle \equiv Vec\,\langle \nabla w_1, \nabla w_2, \nabla w_3\rangle \, , \qquad \forall\, (x,\theta) \in \Omega^0_r \times \mathbb{T}\, . \tag{2.50}$$

The restriction (2.50) means that the three components $w_i$ can be expressed as functions of $\varphi$, $\psi$ and $\theta$. At this level, we recover (2.13).

## 2.3 Necessary and sufficient constraints on $(\varphi, \psi, \mathbf{W})$.

In this Section 2.3, we first show the Proposition 2.3, see the paragraph 2.3.1. Then, in the paragraph 2.3.2, we exclude the situations already examined in [5] and we precise the assumptions on $(\varphi, \psi, \mathbf{W})$ to be retained.

### 2.3.1 The proof of the Proposition 2.3.

The restriction (2.17) is just a repetition of (2.1). Concerning (2.18), it comes from the constraint (2.2) in which the matrix $D_x w(x,\theta)$ is replaced as in (2.14). The relation (2.1) induces simplifications leading to (2.18). With (2.14), we can formulate (2.3) according to

$$(D_x w)^2\, \partial_\varphi \mathbf{W} \otimes \nabla\varphi \, + \, (D_x w)^2\, \partial_\psi \mathbf{W} \otimes \nabla\psi \, = \, 0\, .$$

Recall (2.15). The two vectors $\nabla\varphi$ and $\nabla\psi$ being independent, the above identity is equivalent to

$$(D_x w)^2\, \partial_\varphi \mathbf{W} \;\; = \;\; 0\, , \tag{2.51}$$

$$(D_x w)^2\, \partial_\psi \mathbf{W} \;\; = \;\; 0\, . \tag{2.52}$$

Plug (2.14) into (2.51). Then, exploit (2.15) in order to extract

$$(\nabla\varphi \cdot \partial_\varphi \mathbf{W})^2 + (\nabla\psi \cdot \partial_\varphi \mathbf{W})\,(\nabla\varphi \cdot \partial_\psi \mathbf{W}) \;\; = \;\; 0\, , \tag{2.53}$$

$$(\nabla\psi \cdot \partial_\varphi \mathbf{W})\; (\nabla\psi \cdot \partial_\psi \mathbf{W} + \nabla\varphi \cdot \partial_\varphi \mathbf{W}) \;\; = \;\; 0\, . \tag{2.54}$$

We do the same with (2.52). This time, we get

$$(2.55) \qquad (\nabla\psi \cdot \partial_\psi \mathbf{W})^2 + (\nabla\psi \cdot \partial_\varphi \mathbf{W})\,(\nabla\varphi \cdot \partial_\psi \mathbf{W}) \;=\; 0\,,$$

$$(2.56) \qquad (\nabla\varphi \cdot \partial_\psi \mathbf{W})\;(\nabla\psi \cdot \partial_\psi \mathbf{W} + \nabla\varphi \cdot \partial_\varphi \mathbf{W}) \;=\; 0\,.$$

The relations (2.19) and (2.53) are similar. Observe that we cannot have $\nabla\psi \cdot \partial_\psi \mathbf{W} + \nabla\varphi \cdot \partial_\varphi \mathbf{W} \neq 0$. Indeed, in such a case, (2.53), (2.54), (2.55) and (2.56) would provide

$$\partial_\varphi \mathbf{W} \,\in\, Vec\,\langle\nabla\varphi, \nabla\psi\rangle^\perp\,, \qquad \partial_\psi \mathbf{W} \,\in\, Vec\,\langle\nabla\varphi, \nabla\psi\rangle^\perp\,.$$

In other words, because of (2.15), the two vectors $\partial_\varphi \mathbf{W}$ and $\partial_\psi \mathbf{W}$ of $\mathbb{R}^3$ would be colinear. This is clearly not coherent with (2.15). Therefore, we are sure to have (2.20).

Now, we have to show the opposite implication, that is the *"only if"* part of the Proposition 2.3. Using (2.13) and (2.15), the relations (2.1), (2.2), (2.11) and (1.9) are respectively equivalent to (2.17), (2.18), (2.15) and (2.16).

In addition, we have seen that looking at (2.3) is the same as imposing (2.53), (2.54), (2.55) and (2.56). The three conditions (2.53), (2.54) and (2.56) are taken into account at the level of (2.19) and (2.20). In view of (2.20), the remaining condition (2.55) reduces to (2.53).

It remains to check that the relation (2.4) is indeed a consequence of the constraints of the Proposition 2.3. To this end, use (2.14) in order to identify the different terms of (2.4). With (2.19) et (2.20), we can easily recover $M\,(D_x w)^2 \equiv 0$. Then, we can exploit (2.17) and (2.18) to obtain

$$D_x w\, M\, D_x w + (D_x w)^2\, M = (\nabla\psi \cdot \partial_\theta w)\;(\nabla\varphi \cdot \partial_\varphi \mathbf{W} + \nabla\psi \cdot \partial_\psi \mathbf{W})\;\partial_\psi \mathbf{W} \otimes \nabla\varphi\,.$$

In view of (2.20), we have (2.4). The proof of the Proposition 2.3 is finished.

### 2.3.2 Further adjustments.

Before going further in the analysis, we must take care to deal with situations which are not considered in [5]. Noting $\widetilde{\mathbf{W}}(x,\theta) := \mathbf{W}\big(\varphi(x), \psi(x,\theta), \theta\big)$, the article [5] is based on the following condition, see (35) of [5] :

$$(2.57) \qquad \Pi_{\partial_\theta \widetilde{\mathbf{W}}^\perp}\, D_x \widetilde{\mathbf{W}}\, \Pi_{\nabla\varphi^\perp} \equiv \Pi_{(\partial_\theta\psi\, \partial_\psi \mathbf{W} + \partial_\theta \mathbf{W})^\perp}\, (\partial_\psi \mathbf{W} \otimes \nabla\psi)\, \Pi_{\nabla\varphi^\perp} \equiv 0\,.$$

Therefore, in order not to repeat what is made in [5], we have to work with $\mathbf{W}$, $\varphi$ and $\psi$ adjusted such that $\partial_\psi \mathbf{W} \wedge \partial_\theta \mathbf{W} \not\equiv 0$ and $\nabla\psi \wedge \nabla\varphi \not\equiv 0$.

There are different ways to factorize the profile $w(x,\theta)$ as it is proposed in (2.13). Indeed, if $\chi(\varphi,\psi,\theta) \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathbb{T};\mathbb{R})$ is any function such that

$\partial_\psi \chi \not\equiv 0$, noting $\tilde\psi := \chi(\varphi,\psi,\theta)$, we find $w \equiv \mathbf{W}(\varphi,\psi,\theta) \equiv \widetilde{\mathbf{W}}(\varphi,\tilde\psi,\theta)$ with $\mathbf{W}(\varphi,\psi,\theta) \equiv \widetilde{\mathbf{W}}\big(\varphi,\chi(\varphi,\psi,\theta),\theta\big)$. Then, we find $\partial_\psi \mathbf{W} \equiv \partial_\psi \chi \ \partial_{\tilde\psi} \widetilde{\mathbf{W}} \not\equiv 0$ together with

$$\partial_\varphi \mathbf{W} \equiv \partial_\varphi \widetilde{\mathbf{W}} + \partial_\varphi \chi \ \partial_\psi \mathbf{W} / \partial_\psi \chi \,, \qquad \partial_\theta \tilde\psi \equiv \partial_\theta \psi \ \partial_\psi \chi + \partial_\theta \chi \,.$$

In this transformation, the conditions $\partial_\psi \mathbf{W} \not\equiv 0$ and $\partial_\varphi \mathbf{W} \not\equiv 0$ are preserved. On the other hand, we have some freedom concerning $\partial_\theta \psi$. By adjusting $\chi$ conveniently, we can make sure that $\partial_\theta \psi \not\equiv 0$ or $\partial_\theta \psi \equiv 0$. According to circumstances, we will use one or other of these two conditions. In preparation for what follows, we put aside the framework (2.58) given below

$$(2.58) \qquad \partial_\theta \psi \not\equiv 0 \,, \qquad \partial_\varphi \mathbf{W} \not\equiv 0 \,, \qquad \partial_\psi \mathbf{W} \wedge \partial_\theta \mathbf{W} \not\equiv 0 \,, \qquad \nabla \psi \wedge \nabla \varphi \not\equiv 0 \,.$$

# 3 Compatible couples in the case $\nabla\varphi \cdot \partial_\psi \mathbf{W} \equiv 0$.

We discuss here the system (2.17)-(2.18)-(2.19)-(2.20) under the restriction (2.58) and when $\nabla\varphi \cdot \partial_\psi \mathbf{W} \equiv 0$. In other words, we have to deal with (2.13), (2.15) and (2.58) combined with

$$(3.1) \qquad \nabla\varphi \cdot \partial_\theta \mathbf{W} = 0,$$

$$(3.2) \qquad \nabla\varphi \cdot \partial_\varphi \mathbf{W} = 0,$$

$$(3.3) \qquad \nabla\psi \cdot \partial_\psi \mathbf{W} = 0,$$

$$(3.4) \qquad \nabla\varphi \cdot \partial_\psi \mathbf{W} = 0.$$

## 3.1 The foliated structure associated to the phase $\varphi$.

The phase $\varphi$ must here inherit some special structure.

**Lemma 3.1.** *Assume (1.9), (2.15) and (2.58) as well as (3.1), (3.2), (3.3) and (3.4). By restricting $r \in \mathbb{R}_+^*$ and by permuting the coordinates $x_1$, $x_2$, $x_3$ and the components $\partial_1\varphi$, $\partial_2\varphi$, $\partial_3\varphi$, we can find two scalar functions $f \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ adjusted such that*

$$(3.5) \qquad \nabla\varphi(x) \equiv {}^t\big(f\circ\varphi(x), 1, g\circ\varphi(x)\big)\,\partial_2\varphi(x)\,, \qquad \forall\, x \in \Omega_r^0 \,.$$

***Proof of the Lemma 3.1.*** The conditions (3.1) and (3.4) say that the direction $\nabla\varphi$ is parallel to $\partial_\theta \mathbf{W} \wedge \partial_\psi \mathbf{W} \not\equiv 0$. It follows that the direction $\nabla\varphi$ can be viewed as a function of only $(\varphi,\psi,\theta)$. By restricting $r \in \mathbb{R}_+^*$ and by permuting the coordinates $x_1$, $x_2$, $x_3$ and the components $\partial_1\varphi$, $\partial_2\varphi$, $\partial_3\varphi$, we can always recover

$$\nabla\varphi = E(\varphi,\psi,\theta)\ \partial_2\varphi \,, \qquad E(\varphi,\psi,\theta) := {}^t\big(f(\varphi,\psi,\theta), 1, g(\varphi,\psi,\theta)\big) \,.$$

Since the function $\varphi$ does not depend on $\theta$, we must have $\partial_\theta \psi \, \partial_\psi E + \partial_\theta E \equiv 0$. When $\partial_\psi E \equiv 0$, we find also $\partial_\theta E \equiv 0$ so that (3.5) is verified. From now on, we supppose that $\partial_\psi E \not\equiv 0$.

The application $\partial_\theta \psi$ can be represented as a function of only the variables $(\varphi, \psi, \theta)$, say $\partial_\theta \psi = k(\varphi, \psi, \theta)$ with $k \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{T}; \mathbb{R})$. Consider any function $\chi(\varphi, \psi, \theta)$ satisfying $\partial_\psi \chi \not\equiv 0$ and $k \, \partial_\psi \chi + \partial_\theta \chi \equiv 0$. Define $\tilde{\psi} := \chi(\varphi, \psi, \theta)$. We can change the set of independent variables $(\varphi, \psi, \theta)$ into $(\varphi, \tilde{\psi}, \theta)$ to find $E(\varphi, \psi, \theta) \equiv \tilde{E}(\varphi, \tilde{\psi}, \theta)$. Observe that

$$\partial_\theta \big[ E(\varphi, \psi, \theta) \big] \equiv \partial_\theta \psi \, \partial_\psi E + \partial_\theta E \equiv 0 \equiv \partial_\theta \big[ \tilde{E}(\varphi, \tilde{\psi}, \theta) \big] \equiv \partial_\theta \tilde{\psi} \, \partial_{\tilde{\psi}} \tilde{E} + \partial_\theta \tilde{E} \, .$$

By construction, we have $\partial_\theta \tilde{\psi} \equiv 0$. It follows that the function $\tilde{E}$ does not depend on $\theta$. Retain that

$$\nabla \varphi = \tilde{E}(\varphi, \tilde{\psi}) \; \partial_2 \varphi \, , \quad \tilde{E}(\varphi, \tilde{\psi}) := {}^t\big( \tilde{f}(\varphi, \tilde{\psi}), 1, \tilde{g}(\varphi, \tilde{\psi}) \big) \, . \tag{3.6}$$

Since $\partial_\psi E \equiv 0$, we must have $\partial_{\tilde{\psi}} \tilde{E} \not\equiv 0$. Writing $\mathbf{W}(\varphi, \psi, \theta) \equiv \widetilde{\mathbf{W}}(\varphi, \tilde{\psi}, \theta)$, we still have to deal with (3.1)-(3.2)-(3.3)-(3.4) but this time with $\widetilde{\mathbf{W}}$ and $\tilde{\psi}$ in place of $\mathbf{W}$ and $\psi$. We decompose $\widetilde{\mathbf{W}}$ into

$$\widetilde{\mathbf{W}}(\varphi, \tilde{\psi}, \theta) = \alpha \; {}^t(0, -\tilde{g}, 1) + \beta \; {}^t(1, -\tilde{f}, 0) + \gamma \; {}^t(\tilde{f}, 1, \tilde{g}) \tag{3.7}$$

where the three functions $\alpha$, $\beta$ and $\gamma$ depend on $\varphi$, $\tilde{\psi}$ and $\theta$. The condition (3.1) yields $\partial_\theta \gamma \equiv 0$. On the other hand, the restriction (3.4) leads to

$$\partial_{\tilde{\psi}} \gamma \, (\tilde{f}^2 + 1 + \tilde{g}^2) - \alpha \, \partial_{\tilde{\psi}} \tilde{g} - \beta \, \partial_{\tilde{\psi}} \tilde{f} + \gamma \, (\tilde{f} \, \partial_{\tilde{\psi}} \tilde{f} + \tilde{g} \, \partial_{\tilde{\psi}} \tilde{g}) \; \equiv \; 0 \, . \tag{3.8}$$

Taking the derivative of (3.8) with respect to $\theta$, we find

$$\partial_\theta \alpha \, \partial_{\tilde{\psi}} \tilde{g} + \partial_\theta \beta \, \partial_{\tilde{\psi}} \tilde{f} \; \equiv \; 0 \, . \tag{3.9}$$

The symmetry of second derivatives expressed in the form $\partial^2_{13} \varphi \equiv \partial^2_{31} \varphi$ can be traducted according to

$$(-\partial_{\tilde{\psi}} \tilde{g}, \tilde{f} \, \partial_{\tilde{\psi}} \tilde{g} - \tilde{g} \, \partial_{\tilde{\psi}} \tilde{f}, \partial_{\tilde{\psi}} \tilde{f}) \cdot {}^t(\partial_1 \tilde{\psi}, \partial_2 \tilde{\psi}, \partial_3 \tilde{\psi}) \; \equiv \; 0 \, . \tag{3.10}$$

Combining (3.9) and (3.10) with $\partial_{\tilde{\psi}} \tilde{E} \not\equiv 0$, we can deduce that

$$\nabla \tilde{\psi} \cdot \partial_\theta \widetilde{\mathbf{W}} \equiv \partial_\theta \beta \, \partial_1 \tilde{\psi} - (\tilde{f} \, \partial_\theta \beta + \tilde{g} \, \partial_\theta \alpha) \, \partial_2 \tilde{\psi} + \partial_\theta \alpha \, \partial_3 \tilde{\psi} \; \equiv \; 0 \, . \tag{3.11}$$

Recall that $\nabla \varphi \wedge \nabla \tilde{\psi} \not\equiv 0$. Thus, the relations (3.1), (3.3), (3.4) and (3.11) indicate that the two vectors $\partial_\theta \widetilde{\mathbf{W}}$ and $\partial_{\tilde{\psi}} \widetilde{\mathbf{W}}$ are colinear. It follows that $\partial_\theta \mathbf{W} \wedge \partial_\psi \mathbf{W} = \partial_\psi \chi \; \partial_\theta \widetilde{\mathbf{W}} \wedge \partial_{\tilde{\psi}} \widetilde{\mathbf{W}} \equiv 0$. This last information is clearly in contradiction with (2.58).

□

Recall here a basic result (see also [3, 5]) concerning (3.5).

**Lemma 3.2.** *Select three functions $f(\varphi)$, $g(\varphi)$ and $\varphi_{00}(x_2)$ in $\mathcal{C}^1(\mathbb{R};\mathbb{R})$. Then, for $r \in \mathbb{R}_+^*$ small enough, there is a unique expression $\varphi(x) \in \mathcal{C}^1(\Omega_r^0;\mathbb{R})$ satisfying (3.5), that is*

$$\partial_1\varphi - f\circ\varphi\ \partial_2\varphi = 0\,, \qquad \partial_3\varphi - g\circ\varphi(x)\ \partial_2\varphi = 0\,, \qquad \forall\, x\in\Omega_r^0 \tag{3.12}$$

*together with the initial data $\varphi(0,x_2,0) = \varphi_{00}(x_2)$ for all $x_2 \in ]-r,r[$.*

***Proof of the Lemma 3.2.*** The Cauchy problem for the first conservation law involved at the level of (3.12), namely

$$\partial_1\varphi_0 - f\circ\varphi_0\ \partial_2\varphi_0 \ =\ 0\,, \qquad \varphi_0(0,x_2) = \varphi_{00}(x_2) \tag{3.13}$$

has a local $\mathcal{C}^1$ solution $\varphi_0(x_1,x_2)$ near the point $(0,0)\in\mathbb{R}^2$. Then, consider the local $\mathcal{C}^1$ solution $\varphi(x)$ of

$$\partial_3\varphi - g\circ\varphi(x)\ \partial_2\varphi \ =\ 0\,, \qquad \varphi(x_1,x_2,0) = \varphi_0(x_1,x_2)\,. \tag{3.14}$$

To verify (3.5), it suffices now to check that $\Xi := \partial_1\varphi - f\circ\varphi(x)\,\partial_2\varphi \equiv 0$ also when $x_3 \neq 0$. This property is in fact a consequence of the preceding construction which implies that

$$\partial_3\Xi - g\circ\varphi(x)\ \partial_2\Xi \ =\ g'\circ\varphi\ \partial_2\varphi\ \Xi\,, \qquad \Xi(x_1,x_2,0) = 0\,.$$

□

### 3.2 The description of $(\varphi, w)$.

In this paragraph 3.2, the starting point is the description (3.7) which is based on some auxiliary function $\psi(x)$ (not depending on $\theta$). At this stage, we know that $w$ can be put in the form

$$\begin{aligned} w(x,\theta) = \mathbf{W}\big(\varphi(x),\psi(x),\theta\big) &= \alpha\big(\varphi(x),\psi(x),\theta\big)\begin{pmatrix} 0 \\ -g\circ\varphi(x) \\ 1\end{pmatrix} \\ &+ \beta\big(\varphi(x),\psi(x),\theta\big)\begin{pmatrix} 1 \\ -f\circ\varphi(x) \\ 0\end{pmatrix} + \gamma\big(\varphi(x),\psi(x),\theta\big)\begin{pmatrix} f\circ\varphi(x) \\ 1 \\ g\circ\varphi(x)\end{pmatrix}\end{aligned} \tag{3.15}$$

with a phase $\varphi$ satisfying (3.12). It remains to adjust the ingredients $\varphi$, $\psi$ and $\mathbf{W}$ according to (3.1)-$\cdots$-(3.4). We have already observed that the constraint (3.1) is the same as $\partial_\theta\gamma \equiv 0$. In the same way, using again (3.12), the condition (3.4) is equivalent to $\partial_\psi \gamma \equiv 0$. Thus, the function $\gamma$ depends only on the variable $\varphi$. Retain that $\gamma(\varphi, \psi, \theta) \equiv \gamma(\varphi)$.

Now, we can interpret the two remaining restrictions (3.2) and (3.3) into

$$\begin{aligned} &(3.16) \qquad -\alpha\, g' - \beta\, f' + \gamma'\,(f^2+1+g^2) + \gamma\,(f\, f' + g\, g') \;=\; 0\,, \\ &(3.17) \qquad \partial_\psi \alpha\ \nabla\psi \cdot {}^t(0,-g,1) + \partial_\psi \beta\ \nabla \psi \cdot {}^t(1,-f,0) \;=\; 0\,. \end{aligned}$$

From (3.16), it is easy to extract

$$(3.18) \qquad \partial_\theta \alpha\, g' + \partial_\theta \beta\, f' \;\equiv\; 0\,, \qquad \partial_\psi \alpha\, g' \,+\, \partial_\psi \beta\, f' \;=\; 0\,.$$

The discussion about (3.16)-(3.17) is separated in two cases.

### 3.2.1 The case $f' \equiv g' \equiv 0$.

By hypothesis, we have $f \equiv a$ and $g \equiv b$ with $(a,b) \in \mathbb{R}^2$. It follows that

$$(3.19) \qquad \varphi(x) \;=\; \varphi_{00}(a\,x_1 + x_2 + b\,x_3)\,, \qquad \varphi_{00} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})\,.$$

In view of (3.16), we have also $\gamma \equiv c$ for some $c \in \mathbb{R}$. On the other hand, the function $\psi(x)$ can always be put in the form

$$(3.20) \qquad \psi(x) \;=\; \Psi(x_1, x_3, a\,x_1 + x_2 + b\,x_3)\,, \qquad \Psi(X,Y,Z) \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R})\,.$$

Then, the condition (3.17) becomes the following scalar conservation law (implying $Z$ and $\theta$ as parameters)

$$(3.21) \qquad \partial_\psi \beta\big(\varphi_{00}(Z), \Psi, \theta\big)\ \partial_X \Psi \,+\, \partial_\psi \alpha\big(\varphi_{00}(Z), \Psi, \theta\big)\ \partial_Y \Psi \;\equiv\; 0\,.$$

At the level of (3.21), the variables $Z$ and $\theta$ play the part of parameters. Since $\Psi(X,Y,Z)$ does not depend on $\theta \in \mathbb{T}$, we must have (when $\partial_\psi \alpha \not\equiv 0$)

$$(3.22) \qquad \partial_\psi \beta \;=\; \chi(\varphi, \psi)\ \partial_\psi \alpha\,, \qquad \chi \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})\,.$$

The equation (3.21) reduces to

$$(3.23) \qquad \chi\big(\varphi_{00}(Z), \Psi\big)\ \partial_X \Psi \,+\, \partial_Y \Psi \;\equiv\; 0\,.$$

We can sum up the situation when $\nabla\varphi \cdot \partial_\psi \mathbf{W} \equiv 0$ and $f' \equiv g' \equiv 0$ through the following result.

**Proposition 3.1.** *Select any constants* $(a,b,c) \in \mathbb{R}^3$. *Select any smooth functions* $\varphi_{00}(Z)$, $\chi(\varphi,\psi)$ *and* $\alpha(\varphi,\psi,\theta)$. *Choose any solutions* $\beta(\varphi,\psi,\theta)$ *and* $\Psi(X,Y,Z)$ *satisfying respectively (3.22) and (3.23). Define* $\varphi(x)$ *and* $\psi(x)$ *according to (3.19) and (3.20). Consider the function* $w(x,\theta)$ *given by*

$$w = \alpha(\varphi,\psi,\theta) \begin{pmatrix} 0 \\ -b \\ 1 \end{pmatrix} + \beta(\varphi,\psi,\theta) \begin{pmatrix} 1 \\ -a \\ 0 \end{pmatrix} + c \begin{pmatrix} a \\ 1 \\ b \end{pmatrix}. \tag{3.24}$$

*Then, the couple* $(\varphi,w)$ *is compatible.*

Take $\varphi$ as indicated in (3.19). Given any function $m \in \mathcal{C}^1(\mathbb{R}\times\mathbb{T};\mathbb{R})$, define

$$\beta(\varphi,\psi,\theta) := m(\varphi,\theta) + \varphi \int_0^\psi s\ (\partial_\psi \alpha)(\varphi,s,\theta)\ ds\,.$$

Then, we recover (3.21) with $\psi(x) = x_1/\big(1 + x_3\,\varphi(x)\big)$. The vectors $\nabla\varphi$ and $\nabla\psi$ are not colinear. By choosing $m$ and $\alpha$ conveniently, we can obtain

$$\begin{aligned} \partial_\psi \mathbf{W} \wedge \partial_\theta \mathbf{W} &= (\partial_\psi\alpha\ \partial_\theta\beta - \partial_\psi\beta\ \partial_\theta\alpha)\ {}^t(a,1,b) \\ &= \partial_\psi\alpha\ \big(\partial_\theta m - \varphi \textstyle\int_0^\psi \partial_\theta\alpha(\varphi,s,\theta)\ ds\big)\ {}^t(a,1,b) \not\equiv 0\,. \end{aligned}$$

The relation (2.57) is not satisfied. This example shows that the situations considered in Proposition 3.1 may not fall under the scope of [5].

Note that the support in $(X,Y)$ of any non trivial solution $\Psi \not\equiv 0$ of (3.23) cannot be compact. Moreover, when $\chi$ depends in a non linear way on $\psi$, due to the formation of singularities, the construction is valid only *locally*.

### 3.2.2 The case $f' \not\equiv 0$ or $g' \not\equiv 0$.

In view of (3.18), we must have $\partial_\theta \mathbf{W} \wedge \partial_\psi \mathbf{W} \equiv 0$ that implies (2.57). This situation is excluded at the level of (2.58) because it has been treated in [5].

Still, for the sake of completeness, we explain below what happens. We deal with the case $f' \not\equiv 0$, the other situation ($g' \not\equiv 0$) being similar. This time, seek the function $\psi(x)$ in the form

$$\psi(x) = \Psi\big(x_1,x_3,\varphi(x)\big)\,, \qquad \Psi(X,Y,Z) \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R})\,. \tag{3.25}$$

From (3.18), extract $\partial_\psi\beta$ in function of $\partial_\psi\alpha$. Plug the result into (3.17). Due to (2.58), we must have $\partial_\psi\alpha \not\equiv 0$. Thus, it remains

$$\Psi(X,Y,\varphi) = \Psi_0\big(g'(\varphi)\,Y + f'(\varphi)\,X\big)\,, \qquad \Psi_0 \in \mathcal{C}^1(\mathbb{R};\mathbb{R})\,. \tag{3.26}$$

Thus, the variable $\varphi$ being fixed, the function $\Psi$ is constant on lines. Again, its support cannot be compact.

**Proposition 3.2.** *Select functions $f$, $g$, $\gamma$ and $\Psi_0$ in $\mathcal{C}^1(\mathbb{R};\mathbb{R})$ with $f' \not\equiv 0$. By applying the Lemma 3.2, we can construct a phase $\varphi(x)$ which is solution of (3.12). Define the function $\psi(x)$ as it is indicated in (3.25) and (3.26). Given any $\alpha \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{T};\mathbb{R})$ with $\partial_\psi \alpha \not\equiv 0$, define $\beta \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{T};\mathbb{R})$ through the relation (3.16). Finally, consider the expression $w(x,\theta)$ which is given by (3.15) where $\gamma(\varphi,\psi,\theta) \equiv \gamma(\varphi)$.*

*Then, the couple $(\varphi, w)$ is compatible.*

To illustrate the situation under study, we produce some example. Just take $f(\varphi) = \varphi$, $g(\varphi) = \varphi^{-1}$ and $\gamma(\varphi) \equiv 0$. As a solution of (3.5), we can choose

$$\varphi(x) = \frac{1-x_2}{2x_1} + \sqrt{\Big(\frac{1-x_2}{2x_1}\Big)^2 - \frac{x_3}{x_1}}\,.$$

Concerning $\psi$, given any function $\Xi \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R})$, we can take

$$\psi(x) \,\equiv\, \psi(x,\theta) \,=\, \Xi\big(\varphi(x), x_2 + 2\,\varphi(x)\,x_1, x_3 - \varphi(x)^2\,x_1\big)\,.$$

# 4 Compatible couples in the case $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$.

We discuss here the system (2.17)-(2.18)-(2.19)-(2.20) under the restriction (2.58) and when $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$.

**Lemma 4.1.** *Assume $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$. The couple $(\varphi, w)$ with $w$ given by (2.13) is compatible if and only if there exists a function $k(x,\theta)$ such that*

$$\begin{aligned}
&(4.1) & \nabla\varphi \cdot \partial_\theta w \ &\equiv\ 0\,,\\
&(4.2) & \nabla\psi \cdot \partial_\theta w \ &\equiv\ 0\,,\\
&(4.3) & \nabla\varphi \cdot \big(\partial_\varphi \mathbf{W} - k\,\partial_\psi \mathbf{W}\big) \ &\equiv\ 0\,,\\
&(4.4) & \nabla\psi \cdot \big(\partial_\varphi \mathbf{W} - k\,\partial_\psi \mathbf{W}\big) \ &\equiv\ 0\,,\\
&(4.5) & (k\ \nabla\varphi + \nabla\psi) \cdot \partial_\varphi \mathbf{W} \ &\equiv\ 0\,,\\
&(4.6) & (k\ \nabla\varphi + \nabla\psi) \cdot \partial_\psi \mathbf{W} \ &\equiv\ 0\,.
\end{aligned}$$

***Proof of the Lemma 4.1.*** The relation (4.1) is a repetition of (2.17). When $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$, the condition (2.18) amounts to the same thing as (4.2). On the other hand, from (2.19) and (2.20), we can extract

$$(\nabla\varphi \cdot \partial_\psi \mathbf{W})\,(\nabla\psi \cdot \partial_\varphi \mathbf{W}) - (\nabla\psi \cdot \partial_\psi \mathbf{W})\,(\nabla\varphi \cdot \partial_\varphi \mathbf{W}) \,\equiv\, 0$$

meaning that the vectors ${}^t(\nabla\varphi\cdot\partial_\psi \mathbf{W}, \nabla\psi\cdot\partial_\psi \mathbf{W})$ and ${}^t(\nabla\varphi\cdot\partial_\varphi \mathbf{W}, \nabla\psi\cdot\partial_\varphi \mathbf{W})$ are colinear. The second one can be obtained by multiplying the first one (which by hypothesis is not equal to zero) by a factor $k$. This is precisely (4.3) and (4.4). From (2.19) or (2.20) with (4.3) and (4.4), we can extract (4.5) and (4.6). Reciprocally, from the informations (4.3), (4.4), (4.5) and (4.6), it is easy to deduce (2.19) and (2.20). $\square$

## 4.1 Reduction of the problem : preliminaries.

The system (4.1)-$\cdots$-(4.6) is not yet in a suitable form.

### 4.1.1 Restatement of the problem.

Since $\partial_\psi \mathbf{W} \not\equiv 0$, by permuting the coordinates, we can always suppose that $\partial_\psi \mathbf{W}_2 \not\equiv 0$, allowing to exchange the variable $\psi$ into $\mathbf{W}_2(\varphi,\psi,\theta)$. After this modification, we have to deal with

$$\mathbf{W}(\varphi,\psi,\theta) = {}^t\big(\mathbf{V}(\varphi,\psi,\theta), \psi, \mathbf{W}_3(\varphi,\psi,\theta)\big)\,, \qquad \mathbf{V} := \mathbf{W}_1\,. \tag{4.7}$$

Recall (2.58) which says in paticular that $\partial_\varphi \mathbf{W} \not\equiv 0$. After permuting the two indices 1 and 3 (if necessary), we can suppose that $\partial_\varphi \mathbf{W}_1 \equiv \partial_\varphi \mathbf{V} \not\equiv 0$. It follows that we can regard $\mathbf{W}_3$ as a function of $(\psi,\mathbf{V},\theta)$. In other words, we can find some function $\mathfrak{L}(\psi,\mathbf{V},\theta) \in \mathcal{C}^1(\mathbb{R}^2\times\mathbb{T};\mathbb{R})$ such that

$$\mathbf{W}(\varphi,\psi,\theta) = {}^t\big(\,\mathbf{V}(\varphi,\psi,\theta)\,,\,\psi\,,\,\mathfrak{L}\big(\psi,\mathbf{V}(\varphi,\psi,\theta),\theta\big)\,\big)\,. \tag{4.8}$$

Using (2.58) together with (4.1), (4.2), (4.3) and (4.4), we can see that the two vectors $\partial_\theta w$ and $\partial_\varphi \mathbf{W} - k\,\partial_\psi \mathbf{W}$ are colinear meaning that there is a function $\beta(x,\theta)$ which is adjusted such that

$$\partial_\varphi \mathbf{W} - \tilde{k}\,\partial_\psi \mathbf{W} = \beta\,\partial_\theta \mathbf{W}, \qquad \tilde{k} := k + \beta\,\partial_\theta\psi\,. \tag{4.9}$$

Knowing (4.8), this information (4.9) becomes

$$\beta\,\partial_\theta \mathfrak{L} \equiv 0\,, \qquad \partial_\varphi \mathbf{V} \,=\, \beta\,\partial_\theta \mathbf{V}\,, \qquad k \,=\, -\,\beta\,\partial_\theta\psi\,. \tag{4.10}$$

Since $\partial_\varphi \mathbf{V} \not\equiv 0$, we must have $\beta \not\equiv 0$ and $\partial_\theta \mathbf{V} \not\equiv 0$, this last condition being also a consequence of (2.58). Necessarily, we must have $\partial_\theta \mathfrak{L} \equiv 0$. Introduce

$$v(x,\theta) \,:=\, \mathbf{V}\big(\varphi(x),\psi(x,\theta),\theta\big)\,, \qquad v \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T};\mathbb{R})\,. \tag{4.11}$$

Note simply $\mathfrak{L}(\psi,v) = \mathfrak{L}(\psi,v)(x,\theta) := \mathfrak{L}\big(\psi(x,\theta),v(x,\theta)\big)$. Similarly, note $\partial_\psi \mathfrak{L}(\psi,v) := \partial_\psi \mathfrak{L}\big(\psi(x,\theta),v(x,\theta)\big)$ and $\partial_v \mathfrak{L}(\psi,v) := \partial_{\mathbf{V}} \mathfrak{L}\big(\psi(x,\theta),v(x,\theta)\big)$. Observe that $\nabla\varphi \cdot {}^t(1,0,\partial_v\mathfrak{L})\,\partial_\theta \mathbf{V} = \nabla\varphi\cdot\partial_\theta w - \nabla\varphi\cdot\partial_\psi \mathbf{W}\,\partial_\theta\psi$. In view of the restriction (4.1), the condition $\partial_\theta\psi \not\equiv 0$ of (2.58) and the hypothesis $\nabla\varphi\cdot\partial_\psi \mathbf{W} \not\equiv 0$, we are sure that $\partial_\theta \mathbf{V} \not\equiv 0$. Retain that

$$\partial_\varphi \mathbf{V} \not\equiv 0\,, \qquad \partial_\theta \mathbf{V} \not\equiv 0\,, \qquad \partial_\theta\psi \not\equiv 0\,, \qquad k \equiv -\,\partial_\theta\psi\;\partial_\varphi \mathbf{V}/\partial_\theta \mathbf{V}\,. \tag{4.12}$$

**Proposition 4.1.** *Assume (2.58), (4.8) and* $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$. *Then, the function* $\mathfrak{L}$ *does not depend on the variable* $\theta \in \mathbb{T}$ *and we have (4.12). Moreover, the system (4.1)-$\cdots$-(4.6) is equivalent to*

$$\partial_\theta v \, \big[ \partial_1 \varphi + \partial_v \mathfrak{L}(\psi,v)\, \partial_3 \varphi \big] + \partial_\theta \psi \, \big[ \partial_2 \varphi + \partial_\psi \mathfrak{L}(\psi,v)\, \partial_3 \varphi \big] \, \equiv \, 0 \, , \tag{4.13}$$

$$\partial_\theta v \, \big[ \partial_1 \psi + \partial_v \mathfrak{L}(\psi,v)\, \partial_3 \psi \big] + \partial_\theta \psi \, \big[ \partial_2 \psi + \partial_\psi \mathfrak{L}(\psi,v)\, \partial_3 \psi \big] \, \equiv \, 0 \, , \tag{4.14}$$

$$\partial_1 \psi + \partial_v \mathfrak{L}(\psi,v)\, \partial_3 \psi - \partial_\theta \psi \, \frac{\partial_\varphi \mathbf{V}}{\partial_\theta \mathbf{V}} \, \big[ \partial_1 \varphi + \partial_v \mathfrak{L}(\psi,v)\, \partial_3 \varphi \big] \, \equiv \, 0 \, , \tag{4.15}$$

*where* $v$ *is given by (4.11) whereas* $\partial_\varphi \mathbf{V}$ *and* $\partial_\theta \mathbf{V}$ *are computed at* $(\varphi,\psi,\theta)$.

In view of (4.11), from (4.13) and (4.14), we can easily deduce that

$$\partial_\theta v \, \big[ \partial_1 v + \partial_v \mathfrak{L}(\psi,v)\, \partial_3 v \big] + \partial_\theta \psi \, \big[ \partial_2 v + \partial_\psi \mathfrak{L}(\psi,v)\, \partial_3 v \big] \, \equiv \, 0 \, . \tag{4.16}$$

***Proof of the Proposition 4.1.*** We have already seen that the function $\mathfrak{L}$ does not depend on the variable $\theta \in \mathbb{T}$ and that the conditions inside (4.12) are verified. By construction, we know also that

$$w(x,\theta) \, = \, {}^t\big( v(x,\theta), \psi(x,\theta), \mathfrak{L}(\psi,v)(x,\theta) \big) \, , \qquad \partial_\theta w \not\equiv 0 \, . \tag{4.17}$$

Taking into account (2.58) and (4.8), the two constraints (4.1) and (4.2) are equivalent to the existence of some (nonzero) function $\alpha(x,\theta)$ such that

$$\partial_\theta v \begin{pmatrix} 1 \\ 0 \\ \partial_v \mathfrak{L} \end{pmatrix} + \partial_\theta \psi \begin{pmatrix} 0 \\ 1 \\ \partial_\psi \mathfrak{L} \end{pmatrix} \, = \, \alpha \, \nabla\varphi \wedge \nabla\psi \, . \tag{4.18}$$

Since $\partial_\varphi \mathbf{V} \not\equiv 0$, we have $\partial_\varphi \mathbf{W} \wedge \partial_\psi \mathbf{W} = \partial_\varphi \mathbf{V} \, {}^t(-\partial_v \mathfrak{L}, -\partial_\psi \mathfrak{L}, 1) \not\equiv 0$. Combining this with (4.5), (4.6) and (4.12) yields the existence of some (nonzero) scalar function $\gamma(x,\theta)$ such that

$$- \, \partial_\theta \psi \, \frac{\partial_\varphi \mathbf{V}\big(\varphi(x), \psi(x,\theta), \theta\big)}{\partial_\theta \mathbf{V}\big(\varphi(x), \psi(x,\theta), \theta\big)} \, \nabla\varphi + \nabla\psi \, = \, \gamma \begin{pmatrix} \partial_v \mathfrak{L} \\ \partial_\psi \mathfrak{L} \\ -1 \end{pmatrix} . \tag{4.19}$$

Plug the expression $\nabla\psi$ given by (4.19) into (4.18) in order to extract

$$\partial_\theta v \, = \, - \, \alpha \, \gamma \, (\partial_2 \varphi + \partial_\psi \mathfrak{L} \, \partial_3 \varphi) \, , \tag{4.20}$$

$$\partial_\theta \psi \, = \, + \, \alpha \, \gamma \, (\partial_1 \varphi + \partial_v \mathfrak{L} \, \partial_3 \varphi) \, , \tag{4.21}$$

$$\partial_\theta v \, \partial_v \mathfrak{L} + \partial_\theta \psi \, \partial_\psi \mathfrak{L} \, = \, + \, \alpha \, \gamma \, (\partial_\psi \mathfrak{L} \, \partial_1 \varphi - \partial_v \mathfrak{L} \, \partial_2 \varphi) \, . \tag{4.22}$$

Since $\alpha \, \gamma \not\equiv 0$ (because $\partial_\theta \psi \not\equiv 0$), from (4.20) and (4.21), we can deduce (4.13). On the other hand, the relation (4.22) provides no new information because it is a linear combination of (4.20) and (4.21). Observe that

$$(1,0,\partial_v \mathfrak{L}) \cdot {}^t(\partial_v \mathfrak{L}, \partial_\psi \mathfrak{L}, -1) \equiv 0 \, , \qquad (0,1,\partial_\psi \mathfrak{L}) \cdot {}^t(\partial_v \mathfrak{L}, \partial_\psi \mathfrak{L}, -1) \equiv 0 \, .$$

Using these identities and (4.13), coming back to (4.19) multiplied by the non zero vector valued function $\partial_\theta w$, we can obtain (4.14). The last condition (4.15) is just the product of (4.19) with the vector ${}^t(1, 0, \partial_v \mathfrak{L})$.

Conversely, suppose that $\varphi(x)$ and $\psi(x, \theta)$ are such that $\nabla\varphi \wedge \nabla\psi \not\equiv 0$ and satisfy (locally) the system (4.13)-(4.14)-(4.15) for some functions $\mathfrak{L}(\psi, v)$ and $\mathbf{V}(\varphi, \psi, \theta)$. Define $v$ and $w$ as in (4.11) and (4.17).

Both (4.1) and (4.2) become a direct consequence of (4.13) and (4.14). We can obtain (4.9), that is (4.3) and (4.4), through (4.10) by adjusting the coefficient $\beta$ (and then $k$) conveniently. At this stage, the interpretation of (4.13)-(4.14)-(4.15) is that the vector on the left of (4.19) is orthogonal to the direction ${}^t(1, 0, \partial_v \mathfrak{L})$ and ${}^t(0, 1, \partial_\psi \mathfrak{L})$. Thus, we must have (4.19) for some coefficient $\gamma$. This is exactly (4.5) and (4.6). $\square$

Since $\partial_\theta w \not\equiv 0$, by a small rotation in the space variable $x \in \mathbb{R}^3$, we can always obtain that $\partial_\theta v \not\equiv 0$. All the restrictions in (4.12) are stable under such a modification (if it is small enough). In what follows, we work locally in $(x, \theta)$ under the assumptions $\partial_\theta v \not\equiv 0$ and (4.12). We will exploit these informations in order to perform different changes of variables which are crucial when discussing the content of (4.13)-(4.14)-(4.15).

### 4.1.2 Various changes of variables.

Subtract (4.15) from (4.14), use (4.13) to replace $\partial_1\varphi + \partial_v \mathfrak{L}(\psi, v)\, \partial_3\varphi$, and then exploit (2.58) to make simplifications in order to extract the identity

$$(\partial_2\psi + \partial_\psi \mathfrak{L}\ \partial_3\psi)/\partial_\theta\psi \;=\; \partial_\varphi \mathbf{V}\ (\partial_2\varphi + \partial_\psi \mathfrak{L}\ \partial_3\varphi)/\partial_\theta \mathbf{V}\,. \tag{4.23}$$

The identity $a/b = c/d$ implies that $a/b = (c + \gamma\, a)/(d + \gamma\, b)$ for all $\gamma \in \mathbb{R}$. This implication applied to (4.23) with $\gamma = \partial_\psi \mathbf{V}$ furnishes

$$\frac{\partial_2\psi + \partial_\psi \mathfrak{L}\ \partial_3\psi}{\partial_\theta\psi} \;=\; \frac{\partial_\varphi \mathbf{V}\ (\partial_2\varphi + \partial_\psi \mathfrak{L}\ \partial_3\varphi) + \partial_\psi \mathbf{V}\ (\partial_2\psi + \partial_\psi \mathfrak{L}\ \partial_3\psi)}{\partial_\theta \mathbf{V} + \partial_\psi \mathbf{V}\ \partial_\theta\psi}\,.$$

Recalling (4.11), this is the same as

$$(\partial_2\psi + \partial_\psi \mathfrak{L}\ \partial_3\psi)/\partial_\theta\psi \;=\; (\partial_2 v + \partial_\psi \mathfrak{L}\ \partial_3 v)/\partial_\theta v\,. \tag{4.24}$$

Since $\partial_\theta v \not\equiv 0$, we can work (locally) with the variables $(x, v)$ in place of $(x, \theta)$. The function $\varphi$ does not depend on $\theta \in \mathbb{T}$ and therefore it does not depend on $v$. On the contrary, the function $\psi$ can be put in the form

$$\psi(x, \theta) \;=\; u\big(x, v(x, \theta)\big)\,, \qquad \partial_v u \;\not\equiv\; 0\,. \tag{4.25}$$

Formulating (4.24) at the level of $u(x,v)$ yields

$$\partial_2 u \,+\, \partial_u \mathfrak{L}(u,v)\, \partial_3 u \,=\, 0\,. \tag{4.26}$$

Recalling (4.16) and exploiting (4.26), the constraint (4.14) becomes

$$\partial_1 u \,+\, \partial_v \mathfrak{L}(u,v)\, \partial_3 u \,=\, 0\,. \tag{4.27}$$

Knowing what is the function $u(x,v)$, it is not complicated to obtain $v(x,\theta)$. To this end, it suffices to consider the scalar conservation law

$$\begin{aligned} &\partial_1 v + \partial_v \mathfrak{L}\big(u(x,v),v\big)\; \partial_3 v \\ &\qquad + \partial_v u(x,v)\; \big[\partial_2 v + \partial_u \mathfrak{L}\big(u(x,v),v\big)\; \partial_3 v\big] \;\equiv\; 0\,. \end{aligned} \tag{4.28}$$

To sum up, the system (4.13)-(4.14)-(4.15) amounts to the same thing as to identify the two expressions $u(x,v)$ and $v(x,\theta)$ as it is explained above and then to focus on the remaining constraint, namely

$$\begin{aligned} &\partial_1 \varphi + \partial_v \mathfrak{L}\big(u(x,v),v\big)\; \partial_3 \varphi \\ &\qquad + \partial_v u(x,v)\; \big[\partial_2 \varphi + \partial_u \mathfrak{L}\big(u(x,v),v\big)\; \partial_3 \varphi\big] \;\equiv\; 0\,. \end{aligned} \tag{4.29}$$

Recall that $v \in K \subset \mathbb{R}$ must be seen here, at the level of (4.29), as a parameter. Thus, all the difficulty is *to solve (4.29) with a phase* $\varphi(x)$ *which does not depend on* $v$. We first explain what happens when $\partial_3 u \equiv 0$. Then, we present the problematic when $\partial_3 u \not\equiv 0$.

• **<u>The</u> <u>case</u>** $\boxed{\partial_3 u \equiv 0}$**.** In view of (4.26) and (4.27), we have $\nabla_x u \equiv 0$. It follows that $u(x,v) = U(v)$ with a function $U \in \mathcal{C}^1(K;\mathbb{R})$ such that $U' \not\equiv 0$. Necessarily, the function $\mathbf{V}$ depends only on the variable $\psi$. This is a contradiction with (4.12). For the sake of completeness, we still describe below what happens when $\partial_3 u \equiv 0$ and $U' \not\equiv 0$. Noting $\tilde{\mathfrak{L}}(v) := \mathfrak{L}\big(U(v),v\big)$, we can see that (4.29) becomes

$$\partial_1 \varphi(x) + U'(v)\; \partial_2 \varphi(x) + \tilde{\mathfrak{L}}'(v)\; \partial_3 \varphi(x) \;\equiv\; 0\,. \tag{4.30}$$

Recall that the variables $x$ and $v$ are independent. Thus, the relation (4.30) implies that $\tilde{\mathfrak{L}}(v) \equiv \mathfrak{L}\big(U(v),v\big) \,=\, a\,v + b\,U(v) + c$ for some $(a,b,c) \in \mathbb{R}^3$. We have to deal with

$$\partial_1 \varphi + a\; \partial_3 \varphi + U'(v)\; (\partial_2 \varphi + b\; \partial_3 \varphi) \;\equiv\; 0\,. \tag{4.31}$$

This is possible only if $U' \equiv c \in \mathbb{R}$ and

$$\varphi(x) \,=\, \Phi\big(c\,x_1 - x_2, (a + b\,c)\,x_2 - c\,x_3\big)\,, \qquad \Phi \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R};\mathbb{R})\,.$$

On the other hand, the function $v$ can be obtained through

$$\partial_1 v + a\ \partial_3 v + U'(v)\ (\partial_2 v + b\ \partial_3 v)\ =\ 0\,, \qquad v(0,\cdot) = v_0\,. \tag{4.32}$$

By varying the ingredients $a$, $b$, $U$, $\Phi$ and $v_0$, we can obtain a whole class of solutions to the system (4.13)-(4.14)-(4.15).

• **<u>The</u> <u>case</u>** $\boxed{\partial_3 u \not\equiv 0}$**.** Since $\partial_3 u \not\equiv 0$, we can exchange the variables $(x,v)$ into $(x_1,x_2,u,v)$. In particular, the applications $\varphi$, $\partial_v u$ and $\partial_3 u$ can be regarded as functions of $(x_1,x_2,u,v)$ instead of $(x,v)$. Taking this point of view into account, we adopt the following conventions

$$\varphi(x)\ =\ \Phi\big(x_1,x_2,u(x,v),v\big)\,, \qquad \forall\,(x,v)\,, \tag{4.33}$$

$$\partial_v u(x,v) = R\big(x_1,x_2,u(x,v),v\big)\,, \qquad \forall\,(x,v)\,, \tag{4.34}$$

$$\partial_3 u(x,v) = S\big(x_1,x_2,u(x,v),v\big)\,, \qquad \forall\,(x,v)\,. \tag{4.35}$$

Recall that $R \not\equiv 0$ and $S \not\equiv 0$. The constraint (4.29) becomes

$$X\ \Phi\ \equiv\ 0\,, \qquad X := \partial_1 + R\ \partial_2\,, \tag{4.36}$$

whereas the fact that $\varphi$ does not depend on $v$ amounts to the same thing as

$$Y\ \Phi\ \equiv\ 0\,, \qquad Y := R\ \partial_u + \partial_v\,. \tag{4.37}$$

The rest of this chapter 4 is devoted to the case $\partial_3 u \not\equiv 0$. Thus, it should be clearly noted here what the current matter is.

**Remaining work.** *When $\partial_3 u \not\equiv 0$, the problem is to find a <u>non</u> <u>constant</u> function $\Phi(x_1,x_2,u,v)$ satisfying the transport equations (4.36) and (4.37) with a coefficient $R(x_1,x_2,u,v)$ issued from (4.26), (4.27) and (4.34).*

Forcing the presence of $u$ and $v$ at the level of $\varphi$ and passing through (4.37) to express that $\partial_\theta \varphi \equiv 0$ may seem unnatural. However, this process allows to simplify the equation (4.29). It leads to the above problem which, to our knowledge, is original. The strategy to solve it is the following.

In the Section 4.2, we extract from (4.36)-(4.37) the necessary and sufficient conditions (4.39) and (4.40) to impose on $R$. In the Section 4.3, we exhibit the special form (4.51) of a coefficient $R$ coming from (4.26), (4.27) and (4.34). In the Section 4.4, we test our criteria (4.39) and (4.40) on the functions $R$ which conform to (4.51). All requirements are met in different cases leading to a classification of all compatible couples (when $\nabla\varphi\cdot\partial_\psi \mathbf{W} \not\equiv 0$ and $\partial_3 u \not\equiv 0$). Illustrative examples are proposed in the Section 4.6.

## 4.2 Reduction of the problem : geometrical step.

The existence of a non constant solution to (4.36)-(4.37) relies deeply on the geometrical properties of the two vector fields $X$ and $Y$. Introduce the Lie algebra $\mathcal{A}$ generated by the successive Poisson brackets of $X$ and $Y$. The dimension being 4, we find here

$$\mathcal{A} \equiv \big\langle X \, , \, Y \, , \, [X;Y] \, , \, \big[X;[X;Y]\big] \, , \, \big[Y;[X;Y]\big] \, \big\rangle \, .$$

**Proposition 4.2.** *The system (4.36)-(4.37) has a non constant solution $\Phi$ if and only if the dimension of $\mathcal{A}$ is strictly less than 4. Two different situations may occur :*

**i) dim $\mathcal{A} = 2$.** *The function $\Phi(x_1, x_2, u, v)$ depends on two independent variables. Then, the coefficient $R$ must satisfy :*

$$\partial_1 R + R \, \partial_2 R \, \equiv \, X \, R \, \equiv \, 0 \, , \qquad R \, \partial_u R + \partial_v R \, \equiv \, Y \, R \, \equiv \, 0 \, . \tag{4.38}$$

**ii) dim $\mathcal{A} = 3$.** *The function $\Phi(x_1, x_2, u, v)$ depends on one variable. The coefficient $R$ must satisfy $XR \not\equiv 0$ or $YR \not\equiv 0$ together with*

$$(XR) \; YXR - 2 \; (XR) \; XYR + (YR) \; X^2 R \, = \, 0 \, , \tag{4.39}$$

$$(YR) \; XYR - 2 \; (YR) \; YXR + (XR) \; Y^2 R \, = \, 0 \, . \tag{4.40}$$

***Proof of Proposition 4.2.*** Recall that the Poisson bracket of the vector fields $X$ and $Y$ is the vector field $[X;Y]$ which is adjusted such that

$$[X;Y] \, f = - \, Y R \; \partial_2 f + X R \; \partial_u f = XY f - YX f \, , \qquad \forall \, f \in \mathcal{C}^\infty(\mathbb{R}^4;\mathbb{R}) \, .$$

From (4.36) and (4.37), it is easy to infer that $Z \, \Phi \equiv 0$ for all $Z \in \mathcal{A}$. Thus, when $dim \, \mathcal{A} = 4$, the function $\Phi$ is constant. It means that the phase $\varphi$ is stationary, in contradiction with (1.4). We examine the other situations.

**i) dim $\mathcal{A} = 2$.** This situation can occur if and only if $[X;Y]$ is a linear combination of $X$ and $Y$, giving rise to (4.38). By applying the Frobenius Theorem [1], we see that the field of planes $Vec \, \langle X, Y \rangle$ is associated with a foliated structure of $\mathbb{R}^4$ by submanifolds of dimension 2 along which $\Phi$ must be constant. Clearly, the function $\Phi$ inherits two degrees of freedom. In particular, it can be a non constant solution of the system (4.36)-(4.37).

**i) dim $\mathcal{A} = 3$.** To avoid (4.38), we have to require $XR \not\equiv 0$ or $YR \not\equiv 0$. Then, to obtain $dim \, \mathcal{A} = 3$, it is necessary to impose

$$\big[X;[X;Y]\big] \in Vec \big\langle X, Y, [X;Y] \big\rangle \, , \tag{4.41}$$

$$\big[Y;[X;Y]\big] \in Vec \big\langle X, Y, [X;Y] \big\rangle \, . \tag{4.42}$$

Under the conditions (4.41) and (4.42), we find that $dim\,\mathcal{A}=3$. By applying again the Frobenius Theorem [1], we can see that the field of hyperplanes $Vec\,\big\langle X, Y, [X;Y]\big\rangle$ is associated with a foliated structure of the space $\mathbb{R}^4$ by hypersurfaces along which $\Phi$ must be constant. On the other hand, the function $\Phi$ can actually vary in the directions which are transversal to these hypersurfaces. Now, it remains to convert (4.41)-(4.42) in the form of constraints implying the coefficient $R$. To this end, compute

$$\begin{aligned}\big[X;[X;Y]\big]\,f \,&=\, (-\,2\,XYR+YXR)\;\partial_2 f \,+\, X^2R\;\partial_u f\,,\\ \big[Y;[X;Y]\big]\,f \,&=\, -\,Y^2R\;\partial_2 f \,+\, (2\,YXR-XYR)\;\partial_u f\,.\end{aligned}$$

Taking into acount these informations combined with the specific forms of $X$, $Y$ and $[X;Y]$, we can deduce that the two constraints (4.41) and (4.42) can be verified on condition that $[X;Y]$ is colinear to both $\big[X;[X;Y]\big]$ and $\big[Y;[X;Y]\big]$. This remark, leads directly to (4.39) and (4.40). $\square$

Given some initial data $R(0,x_2,u,0):=R_{00}(x_2,u)$, we can solve the system of two conservation laws (4.38) in the same way as in the Lemma 3.2. Then, to recover $\Phi$, it suffices to fix any function $\Phi_{00}(x_2,u)$ satisfying $\nabla_{x_2,u}\Phi_{00}\not\equiv 0$ and to integrate the two equations

$$\partial_1\Phi + R\;\partial_2\Phi \;\equiv\; 0\,,\qquad R\;\partial_u\Phi + \partial_v\Phi \;\equiv\; 0\,. \tag{4.43}$$

The discussion about (4.39)-(4.40) is delicate. We explain below how to construct $R$ and $\Phi$ in the more general situation (when $XR\;YR\not\equiv 0$).

**Lemma 4.2.** *Fix any (non zero) function $\mathcal{Q}(y,R,\Phi)\in\mathcal{C}^3(\mathbb{R}^3;\mathbb{R}^*)$. Select any couple of functions $R_{00}(x_1,x_2)\in\mathcal{C}^1(\mathbb{R}^2;\mathbb{R})$ and $\Phi_{00}(x_1,x_2)\in\mathcal{C}^1(\mathbb{R}^2;\mathbb{R})$ satisfying $\nabla_{x_1,x_2}\Phi_{00}\not\equiv 0$ as well as*

$$\partial_1 R_{00} + R_{00}\;\partial_2 R_{00}\;\not\equiv\;0\,,\qquad \partial_1\Phi_{00} + R_{00}\;\partial_2\Phi_{00}\;\equiv\;0\,. \tag{4.44}$$

*Then, the system (4.39)-(4.40) has a solution $R(x_1,x_2,u,v)$ such that*

$$R(x_1,x_2,0,0)=R_{00}(x_1,x_2)\,,\qquad X\,R\not\equiv 0\,,\qquad Y\,R\not\equiv 0\,. \tag{4.45}$$

*Moreover, there exists a non constant solution $\Phi$ of (4.36)-(4.37) such that*

$$\Phi(x_1,x_2,0,0)=\Phi_{00}(x_1,x_2)\,,\qquad \frac{Y\,R}{X\,R}\equiv\mathcal{Q}(R\,x_1-x_2,R,\Phi)\,. \tag{4.46}$$

***Proof of Lemma 4.2.*** We start by studying a little more the structure of the system (4.39)-(4.40). Since $XR \not\equiv 0$, we can introduce the quantity $Q := YR/XR$. In fact, the restrictions (4.39) and (4.40) are equivalent to

$$YQ - Q\,XQ \,=\, 0\,, \tag{4.47}$$

$$-\,[Y;X]\,R + (XQ)\;(XR) \,=\, 0\,. \tag{4.48}$$

Since $YR \not\equiv 0$ whereas $X\Phi \equiv 0$, we can always consider that $Q$ is a function of the variables $(x_1, x_2, R, \Phi)$, namely

$$Q(x_1,x_2,u,v) \,=\, \mathfrak{Q}\big(x_1,x_2,R(x_1,x_2,u,v),\Phi(x_1,x_2,u,v)\big)\,.$$

In view of the definition of $Q$ and knowing that $X\Phi \equiv 0$ and $Y\Phi \equiv 0$, the equation (4.47) reduces to $X\mathfrak{Q} = 0$, meaning that $\mathfrak{Q} \equiv \mathcal{Q}(R\,x_1 - x_2, R, \Phi)$ for some function $\mathcal{Q}(T,R,\Phi) \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R})$.

The conditions (4.47) and (4.48) become the two scalar conservation laws

$$\begin{aligned} \partial_v R \,+\, R\,\partial_u R - \,\mathcal{Q}(R\,x_1 - x_2,R,\Phi)\;\partial_1 R \qquad & \\ -\,\mathcal{Q}(R\,x_1 - x_2,R,\Phi)\;R\;\partial_2 R \,\equiv\, 0\,, & \end{aligned} \tag{4.49}$$

$$\begin{aligned} \partial_u R - \,\mathcal{Q}(R\,x_1 - x_2,R,\Phi)\;\partial_2 R \qquad\qquad & \\ +\,(x_1\,\partial_T \mathcal{Q} + \partial_R \mathcal{Q})(R\,x_1 - x_2,R,\Phi)\;(\partial_1 R + R\,\partial_2 R) \,\equiv\, 0\,. & \end{aligned} \tag{4.50}$$

Consider the equation (4.50) written for $R_0(x_1,x_2,u)$ and associated with the initial data $R_0(x_1,x_2,0) = R_{00}(x_1,x_2)$. At first sight, the access to $R_0$ (and $R$) requires the knowledge of $\Phi_0(x_1,x_2,u) := \Phi(x_1,x_2,u,0)$ (and $\Phi$). Nevertheless, by construction, the function $\Phi$ is constant along the characteristics associated with (4.49) and (4.50). Thus, in doing so, it suffices to know who is $\Phi_{00}(x_1,x_2) := \Phi_0(x_1,x_2,0)$.

Look at (4.49) as an evolution equation in $v$ associated with the initial data $R_0$. For the same reasons as above, we can solve this Cauchy problem knowing only who is $\Phi_{00}$. There is still a difficulty coming from a problem of compatibility between (4.49) and (4.50). We must check that the expression $R$ thus obtained is still a solution of (4.50). To this end, it suffices to show that (4.50) is propagated (in the direction $v$). This is due to the identity

$$(Y - \alpha\,X)\big\{-[Y;X]\,R + (XQ)\,(XR)\big\} \,=\, \frac{2\,\big\{-[Y;X]\,R + (XQ)\,(XR)\big\}^2}{XR}\,.$$

Note that $Y\,R \not\equiv 0$ as a consequence of (4.49) and $\mathcal{Q} \not\equiv 0$. The function $\Phi$ can be obtained by the same procedure, by first integrating (4.50) and then by looking at (4.49). Geometrically, we have

$$\nabla\Phi \,=\, \lambda\;{}^t(-R\,XR, XR, YR, -R\,YR)\,, \qquad \lambda \neq 0$$

implying that the level surfaces of $\Phi$ intersect the plane $\{u = v = 0\} \subset \mathbb{R}^4$ transversally. Thus, there is a unique function $\Phi$ satisfying (4.46). $\square$

### 4.3 Reduction of the problem : analytical step.

In the preceding paragraph 4.2, we have developped only the aspects of $R$ related to (4.38) or (4.39) and (4.40). However, the coefficient $R(x_1, x_2, u, v)$ is also linked through the implicit relation (4.34) to the selection of a function $u(x, v)$ satisfying (4.26) and (4.27).

At the level of (4.26) and (4.27), the variable $v$ plays the part of a parameter. The situation is the same as in the Lemma 3.2. It suffices to select some data $u_{00}(x_3, v) \equiv u(0, 0, x_3, v)$ such that $\partial_3 u_{00} \not\equiv 0$ in order to obtain (locally in $\mathbb{R}^4$) some solution $u$ of (4.26) and (4.27) satisfying $\partial_3 u \not\equiv 0$.

**Proposition 4.3.** *Let $u(x, v)$ be any (local) solution of (4.26) and (4.27) satisfying $\partial_3 u \not\equiv 0$. Define $R(x_1, x_2, u, v)$ through (4.34). Then, there is a function $\mathfrak{K} \in \mathcal{C}^1(\mathbb{R}^2; \mathbb{R})$ such that $R$ can be put in the form*

$$(4.51) \qquad R(x_1, x_2, u, v) \,=\, -\,\partial_v \alpha(x_1, x_2, u, v) \,/\, \partial_u \alpha(x_1, x_2, u, v)$$

*where the scalar function $\alpha$ is given by*

$$(4.52) \qquad \alpha(x_1, x_2, u, v) \,:=\, \mathfrak{K}(u, v) + \partial_v \mathfrak{L}(u, v)\ x_1 + \partial_u \mathfrak{L}(u, v)\ x_2\,.$$

In this context, the two restrictions $R \not\equiv 0$ and $S \not\equiv 0$ which are prerequisites in the analysis, see after (4.35), become $\partial_u \mathfrak{K} \not\equiv 0$ and $\partial_v \mathfrak{K} \not\equiv 0$.

***Proof of Proposition 4.3.*** From (4.26) and (4.27), it is easy to deduce

$$(4.53) \qquad \partial_1(\partial_v u) + \partial_v \mathfrak{L}\ \partial_3(\partial_v u) + (\partial^2_{vv} \mathfrak{L} + \partial^2_{uv} \mathfrak{L}\ \partial_v u)\ \partial_3 u \,=\, 0\,,$$

$$(4.54) \qquad \partial_1(\partial_3 u) + \partial_v \mathfrak{L}\ \partial_3(\partial_3 u) + \partial^2_{uv} \mathfrak{L}\ (\partial_3 u)^2 \,=\, 0\,,$$

$$(4.55) \qquad \partial_2(\partial_v u) + \partial_u \mathfrak{L}\ \partial_3(\partial_v u) + (\partial^2_{uv} \mathfrak{L} + \partial^2_{uu} \mathfrak{L}\ \partial_v u)\ \partial_3 u \,=\, 0\,,$$

$$(4.56) \qquad \partial_2(\partial_3 u) + \partial_u \mathfrak{L}\ \partial_3(\partial_3 u) + \partial^2_{uu} \mathfrak{L}\ (\partial_3 u)^2 \,=\, 0\,.$$

Since $\partial_3 u \not\equiv 0$, these equations can be interpreted in the variables $x_1$, $x_2$, $u$ and $v$. Then, it remains the following ODEs (with respect to $x_1$ and $x_2$) :

$$(4.57) \qquad \partial_1(R/S) \,=\, -\,\partial^2_{vv} \mathfrak{L}\,, \qquad \partial_1(1/S) \,=\, \partial^2_{uv} \mathfrak{L}\,,$$

$$(4.58) \qquad \partial_2(R/S) \,=\, -\,\partial^2_{uv} \mathfrak{L}\,, \qquad \partial_2(1/S) \,=\, \partial^2_{uu} \mathfrak{L}\,.$$

Observe that $u$ and $v$ play the part of parameters. It is easy to integrate (4.57) and (4.58). There are functions $k(u,v)$ and $h(u,v)$ such that

$$R/S \,=\, k(u,v) - \partial^2_{vv}\mathfrak{L}(u,v)\ x_1 - \partial^2_{uv}\mathfrak{L}(u,v)\ x_2\,, \tag{4.59}$$

$$1/S \,=\, h(u,v) + \partial^2_{uv}\mathfrak{L}(u,v)\ x_1 + \partial^2_{uu}\mathfrak{L}(u,v)\ x_2\,. \tag{4.60}$$

In fact, the two functions $k$ and $h$ are linked together. This is due to the equality of the mixed partials derivatives $\partial_v(\partial_3 u)$ and $\partial_3(\partial_v u)$:

$$\begin{aligned}\partial_v\big[\partial_3 u(x,v)\big] \,&=\, \partial_v\big[S(x_1,x_2,u,v)\big] \,=\, \partial_u S\ R + \partial_v S\\ &=\, \partial_3\big[\partial_v u(x,v)\big] \,=\, \partial_3\big[R(x_1,x_2,u,v)\big] \,=\, \partial_u R\ S\,.\end{aligned}$$

In other words, we must have

$$(R\ \partial_u S - S\ \partial_u R)/S^2 \,=\, -\,\partial_u(R/S) \,=\, -\,\partial_v S/S^2 \,=\, \partial_v(1/S)\,.$$

Apply this at the level of (4.59) and (4.60) to obtain $-\partial_u k = \partial_v h$. There is $\mathfrak{K}(u,v)$ such that $k = -\partial_v\mathfrak{K}$ and $h = \partial_u\mathfrak{K}$. Dividing (4.59) by (4.60) and replacing $k$ and $h$ as indicated previously, we get (4.51) and (4.52). $\square$

The explicit formulas (4.51) and (4.52) indicate that $R = -\partial_1\beta/\partial_2\beta$ with

$$\beta(x_1,x_2,u,v) := \partial_v\mathfrak{K}\ x_1 + \partial_u\mathfrak{K}\ x_2 + \tfrac{1}{2}\ \partial^2_{vv}\mathfrak{L}\ x_1^2 + \partial^2_{uv}\mathfrak{L}\ x_1\ x_2 + \tfrac{1}{2}\ \partial^2_{uu}\mathfrak{L}\ x_2^2\,.$$

Combining the informations obtained in this paragraph 4.3 with (4.36) and (4.37), we can observe that

$$R \,=\, -\,\frac{\partial_1\Phi}{\partial_2\Phi} \,=\, -\,\frac{\partial_1\beta}{\partial_2\beta}\,, \qquad R \,=\, -\,\frac{\partial_v\alpha}{\partial_u\alpha} \,=\, -\,\frac{\partial_v\Phi}{\partial_u\Phi}\,. \tag{4.61}$$

Now, we can produce another interpretation of *the intermediate problem under study* which is emphasized in the introduction.

**Remark 4.3.1.** *The question is to know if we can find two functions $\mathfrak{K}(u,v)$ and $\mathfrak{L}(u,v)$ allowing a simultaneous factorization of some $\Phi$ in the form*

$$\Phi = \mathcal{A}\big(x_1,x_2,\alpha(x_1,x_2,u,v)\big) = \mathcal{B}\big(u,v,\beta(x_1,x_2,u,v)\big)\,, \qquad \nabla\Phi \not\equiv 0\,.$$

Since $\nabla\Phi \not\equiv 0$, the two functions $\mathcal{A}$ and $\mathcal{B}$ cannot be constant. This is the source of the difficulty.

## 4.4 Test of the integrability conditions.

At this stage, we have to plug the coefficient $R$ given by (4.51)-(4.52) into the integrability conditions (4.38) or (4.39)-(4.40). In this procedure, we have a little freedom coming from the choice of $\mathfrak{L}$ and $\mathfrak{K}$. The matter is to check that the related constraints on $\mathfrak{L}$ and $\mathfrak{K}$ can indeed be realized for non trivials choices of $\mathfrak{L}$ and $\mathfrak{K}$.

In the paragraph 4.4.1, we examine the case $dim\,\mathcal{A} = 2$, that is (4.38). Then, in the paragraph 4.4.2, we consider the case $dim\,\mathcal{A} = 3$, that is (4.39)-(4.40).

### 4.4.1 The two-dimensional criterion.

This is when $dim\,\mathcal{A} = 2$. We have to deal with (4.38).

**Lemma 4.3.** *A function $R$ given by (4.51) with $\alpha$ as in (4.52) satisfies (4.38) if and only if one of the two distinct following conditions is verified :*

**i.1.** *We have $\partial^2_{vv}\mathfrak{L} \equiv 0$. The function $\mathfrak{L}$ is linear, say $\mathfrak{L}(u,v) = a + b\,u + c\,v$ with $(a,b,c) \in \mathbb{R}^3$. Moreover, we can find $\mathfrak{R} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ such that*

$$R(u,v) = \mathfrak{R}\big(\mathfrak{K}(u,v)\big)\,, \qquad \mathfrak{R}(\mathfrak{K})\ \partial_u\mathfrak{K} + \partial_v\mathfrak{K} = 0\,. \tag{4.62}$$

**i.2.** *We have $\partial^2_{vv}\mathfrak{L} \not\equiv 0$. We can find $\mathfrak{H} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ such that*

$$\partial_u\mathfrak{L}(u,v) = \mathfrak{H}\big(\partial_v\mathfrak{L}(u,v)\big)\,, \qquad \partial_u\mathfrak{K} - \mathfrak{H}'(\partial_v\mathfrak{L})\ \partial_v\mathfrak{K} = 0\,. \tag{4.63}$$

In the first case (4.62), we are faced with a scalar conservation law. In the second case (4.63), we have to solve some Hamilton-Jacobi equation. In those cases, the determination of $\mathfrak{K}$ and $\mathfrak{L}$ can be achieved once two functions in $\mathcal{C}^1(\mathbb{R};\mathbb{R})$ are given, namely $\mathfrak{R}(\cdot)$ and $\mathfrak{K}(u,0)$ or $\mathfrak{H}(\cdot)$ and $\mathfrak{K}(u,0)$.

***Proof of Lemma 4.3.*** The calculation of $XR$ gives rise to a polynomial fraction in $x$. More precisely, we find $XR = -(\partial_v\alpha)^{-3}\,P(x)$ with

$$P(x) = a_{(0,0)} + \Xi(\mathfrak{L})\ \sum a_\beta\ x^\beta\,, \qquad \Xi(\mathfrak{L}) := \partial^2_{uu}\mathfrak{L}\ \partial^2_{vv}\mathfrak{L} - (\partial^2_{uv}\mathfrak{L})^2\,.$$

The sum runs over all multi-indices $\beta \in \mathbb{N}^2$ such that $1 \leq |\beta| \leq 2$. We find

$$a_{(0,0)} = (\partial_u\mathfrak{K})^2\ \partial^2_{vv}\mathfrak{L} - 2\ \partial_u\mathfrak{K}\ \partial_v\mathfrak{K}\ \partial^2_{uv}\mathfrak{L} + (\partial_v\mathfrak{K})^2\ \partial^2_{uu}\mathfrak{L}\,, \quad a_{(1,0)} = 2\,\partial_v\mathfrak{K}\,,$$
$$a_{(0,1)} = 2\,\partial_u\mathfrak{K}\,, \qquad a_{(2,0)} = \partial^2_{vv}\mathfrak{L}\,, \qquad a_{(1,1)} = 2\ \partial^2_{uv}\mathfrak{L}\,, \qquad a_{(0,2)} = \partial^2_{uu}\mathfrak{L}\,.$$

Suppose that $\Xi(\mathfrak{L}) \not\equiv 0$. Then, the condition $XR \equiv 0$ requires that all the coefficients $a_\beta$ with $|\beta| \leq 2$ are equal to zero. In particular, it follows that $\partial_u\mathfrak{K} \equiv 0$ and $\partial_v\mathfrak{K} \equiv 0$. This is not possible because this situation was excluded. Necessarily, we must impose $\Xi(\mathfrak{L}) \equiv 0$.

**i.1.** When $\partial^2_{vv}\mathfrak{L} \equiv 0$, the condition $\Xi(\mathfrak{L}) \equiv 0$ becomes $\partial^2_{uv}\mathfrak{L} \equiv 0$. It remains $a_{(0,0)} = (\partial_v\mathfrak{K})^2\,\partial^2_{uu}\mathfrak{L} \equiv 0$. The function $\mathfrak{L}$ must be linear in $u$ and $v$, say $\mathfrak{L}(u,v) = a + b\,u + c\,v$. It follows that $R \equiv -\partial_v\mathfrak{K}/\partial_u\mathfrak{K}$. The other constraint $YR \equiv 0$ amounts to the same thing as

$$\partial_u\mathfrak{K}\ (R\ \partial_u R + \partial_v R)\ \equiv\ -\partial_v\mathfrak{K}\ \partial_u R + \partial_u\mathfrak{K}\ \partial_v R\ \equiv\ 0$$

implying that $R \equiv \mathfrak{R}(\mathfrak{K})$ for some $\mathfrak{R} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$. We have (4.62).

**i.2.** When $\partial^2_{vv}\mathfrak{L} \not\equiv 0$, the relation $\Xi(\mathfrak{L}) \equiv 0$ is equivalent to $\partial_u\mathfrak{L} = \mathfrak{H}(\partial_v\mathfrak{L})$ for some $\mathfrak{H} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$. Then, the condition $a_{(0,0)} \equiv 0$ leads to the condition $\partial^2_{vv}\mathfrak{L}\,\big[\partial_u\mathfrak{K} - \mathfrak{H}'(\partial_v\mathfrak{L})\,\partial_v\mathfrak{K}\big]^2 \equiv 0$. We recognize here the second part of (4.63). We find $R \equiv -\mathfrak{H}'(\partial_v\mathfrak{L})^{-1}$ and, combining the preceding informations, it becomes easy to check that the relation $YR \equiv 0$ is sure to be satisfied. $\square$

### 4.4.2 The three-dimensional criterion.

This is when $dim\,\mathcal{A} = 3$. We have to deal with (4.39) and (4.40), knowing that $XR \not\equiv 0$ *or* $YR \not\equiv 0$. We consider separately the different situations which can happen concerning $XR$ or $YR$.

**Lemma 4.4.** *[Case $XR \equiv 0$ and $YR \not\equiv 0$]. A function $R$ given by (4.51) with $\alpha$ as in (4.52) satisfies $XR \equiv 0$, (4.39) and (4.40) without (4.38) if and only if the function $\mathfrak{L}$ is linear, say $\mathfrak{L}(u,v) = a + bu + cv$ with $(a,b,c) \in \mathbb{R}^3$, whereas $R \equiv -\partial_v \mathfrak{K}/\partial_u \mathfrak{K}$ with $\mathfrak{K}$ such that*

$$(\partial_v \mathfrak{K})^2 \ \partial^2_{uu}\mathfrak{K} - 2\ \partial_u \mathfrak{K}\ \partial_v \mathfrak{K}\ \partial^2_{uv}\mathfrak{K} + (\partial_u \mathfrak{K})^2\ \partial^2_{vv}\mathfrak{K} \not\equiv 0\,. \tag{4.64}$$

***Proof of Lemma 4.4.*** The discussion is the same as in the proof of Lemma 4.3. The option **i.2** must be excluded because it leads to $YR \equiv 0$. Just go back to **i.1** where (4.62) must be exchanged with (4.64). $\square$

**Lemma 4.5.** *[Case $XR \not\equiv 0$ and $YR \equiv 0$]. A function $R$ given by (4.51) with $\alpha$ as in (4.52) satisfies $YR \equiv 0$, (4.39) and (4.40) without (4.38) if and only if the function $\mathfrak{K}$ is linear in $u$ and $v$, say $\mathfrak{K}(u,v) = \alpha\, u + \beta\, v + \gamma$ with $\alpha \neq 0$ and $\beta \neq 0$, whereas the function $\mathfrak{L}(u,v)$ is polynomial in $u$ and $v$ with degree less or equal to 2. Moreover, the involved coefficients must be adjusted in order to have $XR \not\equiv 0$.*

***Proof of Lemma 4.5.*** We have (4.40) and the condition (4.39) reduces to $YXR \equiv 0$ yielding $[X;Y]\,R = XYR - YXR \equiv 0 \equiv XR\,\partial_u R$. It means that $\partial_u R \equiv 0$ and therefore $\partial_v R \equiv 0$. The function $R$ does not depend on $(u,v)$. In particular, for $(x_1,x_2) = (0,0)$, we find that $\partial_v \mathfrak{K}/\partial_u \mathfrak{K}$ is constant. Since $\partial_u \mathfrak{K} \not\equiv 0$ and $\partial_v \mathfrak{K} \not\equiv 0$, we must have

$$\mathfrak{K}(u,v) \equiv K(u - a\,v)\,, \qquad a \in \mathbb{R}^*\,, \qquad K \in \mathcal{C}^2(\mathbb{R};\mathbb{R})\,, \qquad K' \not\equiv 0\,.$$

Either $K'' \equiv 0$ and all derivatives $D^\beta \mathfrak{L}$ with $|\beta| = 2$ are constant, leading to the description of Lemma 4.5. Or $K'' \not\equiv 0$ and $\mathfrak{L}(u,v) = F(u - a\,v) + \beta\, v$ for some function $F \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ and some constant $\beta \in \mathbb{R}$. Nevertheless, this last case must be excluded. Indeed, it yields $R \equiv -a$ so that $XR \equiv 0$ (in contradiction with the hypothesis $XR \not\equiv 0$). $\square$

The remaining case is when $XR \not\equiv 0$ and $YR \not\equiv 0$.

**Proposition 4.4.** *[Case $XR \not\equiv 0$ and $YR \not\equiv 0$]. A function $R$ which is such that $(XR)\,(YR) \not\equiv 0$ and which is given by (4.51) with $\alpha$ as in (4.52) satisfies*

*(4.39) and (4.40) when the expressions $\mathfrak{K}$ and $\mathfrak{L}$ are adjusted according to one of the two following (distinct) situations :*

**ii.1.** *Both functions $\mathfrak{L}(u,v)$ and $\mathfrak{K}(u,v)$ are polynomial in $u$ and $v$ with degree less or equal to 2. More precisely, we have*

$$\mathfrak{L}(u,v) = a_{20}\,u^2 + 2\,a_{11}\,u\,v + a_{02}\,v^2 + a_1 u + a_2 v + a_0\,, \quad a_\star \in \mathbb{R}\,, \tag{4.65}$$

$$\mathfrak{K}(u,v) = k_{20}\,u^2 + 2\,k_{11}\,u\,v + k_{02}\,v^2 + k_1 u + k_2 v + k_0\,, \quad k_\star \in \mathbb{R}\,, \tag{4.66}$$

*with coeffecients $a_{20}$, $a_{11}$ and $a_{02}$ (not all equal to zero) and coeffecients $k_{20}$, $k_{11}$ and $k_{02}$ (not all equal to zero) adjusted such that*

$$k_{11}\,a_{02} - k_{02}\,a_{11} = k_{20}\,a_{02} - k_{02}\,a_{20} = k_{20}\,a_{11} - k_{11}\,a_{20} = 0\,. \tag{4.67}$$

**ii.2.** *The functions $\mathfrak{L}(u,v)$ can be put in the form*

$$\mathfrak{L}(u,v) \,=\, a\ u \,+\, \mathbb{F}(b\,u+v)\,, \qquad (a,b) \in \mathbb{R}^2 \tag{4.68}$$

*where the auxiliary function $\mathbb{F} \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$ satisfies $\mathbb{F}^{(3)} \not\equiv 0$ and the ODE*

$$(\gamma\,s^2 + 2\,\beta\,s + \delta)\ \mathbb{F}^{(3)}(s) \,+\, 3\,(\gamma\,s + \beta)\ \mathbb{F}^{(2)}(s) \,=\, 0\,, \qquad s \in \mathbb{R} \tag{4.69}$$

*with constants $\gamma$, $\beta$ and $\delta$ not all equal to zero. The gradient of $\mathfrak{K}(u,v)$ is adjusted as indicated at the level of (4.75) (with polynomial functions $A$ and $B$ which are defined in the proof).*

**ii.3.** *The function $\mathfrak{L}(u,v)$ can be put in the form*

$$\mathfrak{L}(u,v) \,=\, u\ \mathbb{F}\big(u^{-\,1}\,(v+\alpha)\big) \,+\, \mathbb{G}(u)\,, \qquad \alpha \in \mathbb{R} \tag{4.70}$$

*where the auxiliary functions $\mathbb{F} \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ and $\mathbb{G} \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ satisfy*

$$\mathbb{F}^{(2)}(u) \,\neq\, 0\,, \qquad \delta\ \mathbb{F}^{(2)}(u) \,=\, u^3\ \mathbb{G}^{(2)}(u)\,, \qquad u \in \mathbb{R} \tag{4.71}$$

*with $\delta \in \mathbb{R}^*$. Moreover $\mathfrak{K}(u,v) = \partial_v \mathfrak{L}(u,v)$.*

***Proof of Proposition 4.4.*** Below, we check that the different choices described in the paragraphs **ii.1**, **ii.2** and **ii.3** are convenient. Showing that there are no other possible situations is delicate. This aspect of the discussion is postponed to the Appendix 6. Recall that (4.39)-(4.40) is equivalent to (4.47)-(4.48) or to (4.49)-(4.50). We start by looking at the equation (4.49) which is the same as $Z\,R \equiv 0$ where $Z$ is the vector field

$$Z := Y - \mathcal{Q}(R\,x_1 - x_2, R, \Phi)\,X\,, \qquad X = \partial_1 + R\,\partial_2\,, \qquad Y = R\,\partial_u + \partial_v\,.$$

By construction, we have also

$$(4.72)\qquad Z\,(R\,x_1-x_2)\equiv 0\,,\qquad Z\,\Phi\equiv 0\,,\qquad Z\,\big[\mathcal{Q}(R\,x_1-x_2,R,\Phi)\big]\equiv 0\,.$$

Select $\tilde v\in\mathbb{R}$ near 0. Given $f\in\mathcal{C}^1_l(\mathbb{R}^4;\mathbb{R})$, note $f_{\tilde v}(x_1,x_2,u):=f(x_1,x_2,u,\tilde v)$. For instance, we have $R_{\tilde v}(x_1,x_2,u):=R(x_1,x_2,u,\tilde v)$ and

$$Q_{\tilde v}(x_1,x_2,u)\;:=\;Q(x_1,x_2,u,\tilde v)\;=\;\mathcal{Q}(R_{\tilde v}\,x_1-x_2,R_{\tilde v},\Phi_{\tilde v})\,.$$

We also adopt the following conventions

$$\begin{aligned} d_v f &:= \partial_v\big[f(x_1,x_2,u+R_{\tilde v}\,v,\tilde v+v)\big]\\ &\;= (R_{\tilde v}\,\partial_u f+\partial_v f)(x_1,x_2,u+R_{\tilde v}\,v,\tilde v+v)\,,\end{aligned}$$

$$d_v^2 f:=\partial_v(d_v f)\;=\;(R_{\tilde v}^2\,\partial^2_{uu}f+2\,R_{\tilde v}\,\partial^2_{uv}f+\partial^2_{vv}f)(x_1,x_2,u+R_{\tilde v}\,v,\tilde v+v)\,.$$

To avoid confusions, retain that, in general, we have $d_v^2\not\equiv d_v\circ d_v$. In view of (4.72), the characteristic associated with (4.49) and starting from the point $(x_1,x_2,u,\tilde v)$ is a straight line given by

$$(4.73)\qquad (X_1,X_2,U,V)(v)=(\,x_1-Q_{\tilde v}\,v\,,\;x_2-Q_{\tilde v}\,R_{\tilde v}\,v\,,\;u+R_{\tilde v}\,v\,,\;\tilde v+v\,)\,.$$

The function $R$ must be constant along the characteristics. Expressing this principle in connection with the definitions (4.51)-(4.52) yields

$$(4.74)\qquad d_v\mathfrak{K}+d_v(\partial_v\mathfrak{L})\,x_1+d_v(\partial_u\mathfrak{L})\,x_2-Q_{\tilde v}\;v\;d_v^2\mathfrak{L}\;\equiv\;0\,.$$

• **The situation ii.1.** Observe that, due to (4.65), the three quantities $d_v(\partial_v\mathfrak{L})$, $d_v(\partial_u\mathfrak{L})$ and $d_v^2\mathfrak{L}$ are constant functions. Thus, applying the second order derivative $\partial^2_{vv}$ to the identity (4.74), we can extract

$$\partial^3_{uuu}\mathfrak{K}(U,V)\,R_{\tilde v}^3+3\,\partial^3_{uuv}\mathfrak{K}(U,V)\,R_{\tilde v}^2+3\,\partial^3_{uvv}\mathfrak{K}(U,V)\,R_{\tilde v}+\partial^3_{vvv}\mathfrak{K}(U,V)=0\,.$$

Since the three variables $R_{\tilde v}$, $U$ and $V$ are independent, we must have (4.66). Then, observe that

$$\partial_u R\;=\;-2\;(\partial_u\alpha)^{-1}\;(k_{11}+k_{20}\,R)\,,\qquad \partial_v R\;=\;-2\;(\partial_u\alpha)^{-1}\;(k_{02}+k_{11}\,R)\,,$$

$$\partial_1 R\;=\;-2\;(\partial_u\alpha)^{-1}\;(a_{02}+a_{11}\,R)\,,\qquad \partial_2 R\;=\;-2\;(\partial_u\alpha)^{-1}\;(a_{11}+a_{20}\,R)\,.$$

It follows that

$$Q(x_1,x_2,u,v)\;\equiv\;\mathcal{Q}(R)\;=\;\frac{Y\,R}{X\,R}\;=\;\frac{k_{02}+2\,k_{11}\,R+k_{20}\,R^2}{a_{02}+2\,a_{11}\,R+a_{20}\,R^2}\,.$$

We can work at the level of (4.47)-(4.48). By construction, the condition (4.47) is satisfied. On the other hand, (4.48) becomes

$$\partial_u R\;-\;\mathcal{Q}(R)\;\partial_2 R\;+\;\mathcal{Q}'(R)\;(\partial_1 R+R\;\partial_2 R)\;=\;0\,.$$

This relation amounts to the same thing as

$$(k_{11}\,a_{02}-k_{02}\,a_{11})+(k_{20}\,a_{02}-k_{02}\,a_{20})\;R+(k_{20}\,a_{11}-k_{11}\,a_{20})\;R^2\;=\;0\,.$$

This polynomial function of $R$ is identically zero if and only if the restriction (4.67) is verified.

• **The situation ii.2.** Since $\mathbb{F}^{(2)} \not\equiv 0$, we can introduce

$$A(u,v) \,:=\, \frac{\partial_u \mathfrak{K}(u,v)}{\mathbb{F}^{(2)}(b\,u+v)}\,, \qquad B(u,v) \,:=\, \frac{\partial_v \mathfrak{K}(u,v)}{\mathbb{F}^{(2)}(b\,u+v)}\,. \tag{4.75}$$

With these conventions, the function $R$ can be put in the form

$$R \,=\, -\big(B(u,v)+x_1+b\,x_2\big)\,\big(A(u,v)+b\,x_1+b^2\,x_2\big)^{-1}$$

whereas

$$Q \,=\, \tilde{Q}(R,u,v) \,:=\, \big(\partial_u A\,R^2 + (\partial_v A + \partial_u B)\,R + \partial_v B\big)\,(R\,b+1)^{-2}\,.$$

The condition (4.47) reduces to $\partial_v \tilde{Q} + R\,\partial_u \tilde{Q} = 0$. Taking into account the above specific form of $Q$, we find a fraction in $R$ whose coefficients must be zero. This criterion leads to

$$\partial^2_{uu} A \,=\, 2\,\partial^2_{uv} A + \partial^2_{uu} B \,=\, \partial^2_{vv} A + 2\,\partial^2_{uv} B \,=\, \partial^2_{vv} B \,\equiv\, 0\,. \tag{4.76}$$

Exploiting (4.76), we can obtain

$$A(u,v) \,=\, +\,\alpha\,u\,v - \gamma\,v^2 + a^1_0\,u + a^1_1\,v + a^0\,,$$
$$B(u,v) \,=\, -\,\alpha\,u^2 + \gamma\,u\,v + b^1_0\,u + b^1_1\,v + b^0\,.$$

Look at (4.48) which can also be formulated as $\partial_u R - Q\;\partial_2 R + XQ = 0$. Noting $\mathfrak{D} := A + b\,x_1 + b^2\,x_2$, we find that

$$\mathfrak{D}\,(R\,b+1)\,XQ \,=\, 2\,b\,\partial_v B - \partial_v A - \partial_u B + (b\,\partial_v A + b\,\partial_u B - 2\,\partial_u A)\,R\,.$$

Again, the condition (4.48) becomes a fraction in $R$ whose coefficients must be zero. It follows that

$$-3\,\partial_u A + 2\,b\,\partial_v A + b\,\partial_u B \,=\, 0\,, \tag{4.77}$$
$$-2\,\partial_u B + 3\,b\,\partial_v B - \partial_v A \,=\, 0\,. \tag{4.78}$$

It remains $\alpha = -b\,\gamma$ and

$$A(u,v) \,=\, -\,b\,\gamma\,u\,v - \gamma\,v^2 + b\,(-b^1_0 + 2\,b\,b^1_1)\,u + (-2\,b^1_0 + 3\,b\,b^1_1)\,v + a^0\,.$$

Coming back to (4.75), we have to test the existence of $\mathfrak{K}$ through Clairaut's Theorem. This is guaranteed by (4.69) if we choose $\beta := b^1_0 - b\,b^1_1$ and $\delta = b\,b^0 - a^0$. The remaining restriction on $\gamma$, $\beta$ and $\delta$ comes from the two conditions $XR \not\equiv 0$ and $YR \not\equiv 0$.

• **The situation ii.3.** In this context, the definition of $R$ gives rise to

$$R \,=\, -\,\frac{R_1(x_1,x_2,u,v)}{R_2(x_1,x_2,u,v)} \,:=\, -\,\frac{1 + x_1 + a(u,v)\,x_2}{a(u,v)\,(1+x_1) + b(u,v)\,x_2}\,, \tag{4.79}$$

where we have introduced

$$a(u,v) \,:=\, -\,u^{-1}\,(v+\alpha)\,, \qquad b(u,v) \,=\, u^{-2}\,\big[(v+\alpha)^2+\delta\big]\,. \tag{4.80}$$

We can use the formula given for $R$ in (4.79) to compute

$$Q(x_1,x_2,u,v) \,\equiv\, \frac{YR}{XR} \,\equiv\, \frac{(YR_1)\,R_2 \,-\, (YR_2)\,R_1}{(XR_1)\,R_2 \,-\, (XR_2)\,R_1}\,.$$

From (4.79), we can also extract

$$1+x_1 \,=\, -\,\frac{h_1(u,v,R)}{h_2(u,v,R)}\;x_2 \,:=\, -\,\frac{a(u,v)+R\,b(u,v)}{1+R\,a(u,v)}\;x_2\,.$$

Then, replacing $1+x_1$ accordingly in the expression of $Q$, we can derive

$$Q(x_1,x_2,u,v) \,=\, \mathfrak{Q}(R,x_2,u,v) \,=\, \mathfrak{Q}_1(R,u,v)\;\mathfrak{Q}_2(R,u,v)^{-1}\;x_2$$

where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are only functions of $R$, $u$ and $v$. We find

$$\begin{aligned}
\mathfrak{Q}_1 \,&:=\, (b\,Ya - a\,Yb)\;h_2^2 \,+\, Yb\;h_1\;h_2 \,-\, Ya\;h_1^2\,,\\
\mathfrak{Q}_2 \,&:=\, (b-a^2)\;h_2\;(h_2+R\,h_1)\,.
\end{aligned}$$

Many simplifications occur. It remains $Q=-x_2\,u^{-1}$ allowing to check (4.47) directly. On the other hand, the condition (4.48) reduces to

$$R_1\;(u\,\partial_u + x_2\,\partial_2)\,R_2 \,-\, R_2\;(u\,\partial_u + x_2\,\partial_2)\,R_1 \,+\, R_1\;R_2 \,\equiv\, 0\,.$$

Taking into account the definitions of $R_1$, $R_2$, $a$ and $b$, this last relation becomes obvious to verify. □

## 4.5 Discussion summary.

Up to now, we have described which conditions are needed in order to progress. Our aim here is to explain how to proceed concretely in order to build compatible couples $(\varphi,w)$ in the case $\nabla\varphi\cdot\partial_\psi\mathbf{W}\not\equiv 0$. Select two functions $\mathfrak{L}$ and $\mathfrak{K}$ as it is indicated in the paragraphs 4.4.1 or 4.4.2. In particular, we have $\partial_u\mathfrak{K}\not\equiv 0$ and $\partial_v\mathfrak{K}\not\equiv 0$. Define the coefficient $R(x_1,x_2,u,v)$ through (4.51) and (4.52). Knowing $R$, we have access to $\Phi$. More precisely, when $dim\,\mathcal{A}=2$, the function $\Phi$ is entirely determined by prescribing

$$\Phi_{00}(x_2,u) \,:=\, \Phi(0,x_2,u,0) \,\in\, \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})\,, \qquad \nabla_{x_2,u}\Phi_{00}\not\equiv 0\,. \tag{4.81}$$

On the other hand, when $dim\,\mathcal{A}=3$ (implying that $XR\not\equiv 0$ and $YR\not\equiv 0$), the situation is more restricted. Then the function $\Phi_{00}(x_2,u)$ must in fact depend only on one variable, say $u$. Indeed, it can be obtained by solving

$$-\,YR\;\partial_2\Phi_{00} + XR\;\partial_u\Phi_{00} = 0\,, \qquad \Phi_{00}(0,u)=\Phi_{000}(u)\in\mathcal{C}^1(\mathbb{R};\mathbb{R})\,. \tag{4.82}$$

Once the function $\Phi_{00}(x_2,u)$ is fixed as it is indicated above, we can recover a non stationnary phase $\Phi(x_1,x_2,u,v)$. Now, select any function $\chi \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ such that $\chi' \not\equiv 0$ and consider the solution $u_{00}(x_3,v)$ of the following ordinary differential equation (in the variable $v$)

$$\partial_u \mathfrak{K}(u_{00},v)\ \partial_v u_{00} \,+\, \partial_v \mathfrak{K}(u_{00},v) \,=\, 0\,, \qquad u_{00}(x_3,0)=\chi(x_3)\,. \tag{4.83}$$

By construction, the expression $u_{00}(x_3,v)$ satisfy (4.34). The resolution of the equations (4.26) and (4.27), where $v$ belongs to some compact set $K\subset\mathbb{R}$ and plays the part of a parameter, has already been discussed.

Given $\mathfrak{L}$ and $u_{00}$, there is a unique expression $u(x,v)\in\mathcal{C}^1(\Omega_r^0\times K;\mathbb{R})$ satisfying (4.26)-(4.27) together with the inital data

$$u(0,0,x_3,v) \,=\, u_{00}(x_3,v)\in\mathcal{C}^1(\mathbb{R}^2;\mathbb{R})\,. \tag{4.84}$$

Moreover, as a consequence of the proof of Proposition 4.3, we have the relation (4.34) for all $(x,v)$. Deducing the expression $\varphi$ from $\Phi(x_1,x_2,u,v)$ and $u(x,v)$ through the formula (4.33), we obtain a function $\varphi(x)$ which does not depend on the variable $v$.

The determination of the function $v(x,\theta)$ is delicate. Combining (4.11) and (4.25), we can extract the functional identity

$$v(x,\theta) \,=\, \mathbf{V}\big(\varphi(x),u\big(x,v(x,\theta)\big),\theta\big)\,, \qquad \forall\,(x,\theta)\in\Omega_r^0\times\mathbb{T}\,. \tag{4.85}$$

In particular, for $x=0$ and for all $\theta\in\mathbb{T}$, we are faced with

$$v(0,\theta)=\mathbf{V}\big(\varphi(0),u_{00}\big(0,v(0,\theta)\big),\theta\big)\,, \qquad \varphi(0)=\Phi_{00}\big(0,\chi(0)\big)\,.$$

To simplify, we can seek a function $v(x,\theta)$ such that $v(0,\cdot)\equiv 0$. It means that the function $\mathbf{V}(\varphi,\psi,\theta)$ must be such that

$$\mathbf{V}\big(\varphi(0),\chi(0),\theta\big)\equiv 0\,, \qquad \forall\,\theta\in\mathbb{T}\,. \tag{4.86}$$

In what follows, we select a function $\mathbf{V}$ satisfying (4.86). We suppose also that $\partial_\theta\mathbf{V}$ is not the zero function and that

$$\partial_v\mathfrak{K}\big(\chi(0),0\big)\ \partial_\psi\mathbf{V}\big(\varphi(0),\chi(0),\theta\big)+\partial_u\mathfrak{K}\big(\chi(0),0\big)\,\neq\,0\,, \qquad \forall\,\theta\in\mathbb{T}\,. \tag{4.87}$$

For each $\theta\in\mathbb{T}$, the informations (4.86) and (4.87) allow to apply the implicit Theorem at the point $(0,\theta,0)$ to the application

$$\begin{array}{rcl} \mathbb{R}^3\times\mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ (x,\theta,v) & \longmapsto & v-\mathbf{V}\big(\varphi(x),u(x,v),\theta\big)\,. \end{array}$$

It yields locally, near $(0, \theta) \in \mathbb{R}^3 \times \mathbb{T}$, a unique function $v(x, \theta)$ satisfying the relation of (4.85). Due to the compactness of the torus $\mathbb{T}$, by adjusting the number $r \in \mathbb{R}_*^+$ sufficiently small, we can recover (4.85). Note that the expression $v(x, \theta)$ is (by construction) necessarily a solution of (4.28). Moreover, we do not have $\partial_\theta v \equiv 0$.

Define the function $\psi(x, \theta)$ through (4.25). All the ingredients $\varphi(x)$, $\psi(x, \theta)$ and $\mathbf{V}(\varphi, \psi, \theta)$ are determined. It means that the profile $w(x, \theta)$ is known. Just use (2.13) and (4.8). By construction, the couple $(\varphi, w)$ is compatible. Below, we sum up the preceding discussion by clearly precising the degrees of freedom at disposal in the construction of compatible couples $(\varphi, w)$.

**Proposition 4.5.** *In the case $\nabla\varphi \cdot \partial_\psi \mathbf{W} \not\equiv 0$, the class of compatible couples $(\varphi, w)$ is entirely determined by giving locally*

- *functions $\mathfrak{L}(u, v)$ and $\mathfrak{K}(u, v)$ coming from the paragraphs 4.4.1 or 4.4.2 ;*
- *a function $\Phi_{00}(x_2, u)$ which must satisfy (4.82) when $\dim \mathcal{A} = 3$ ;*
- *a function $\chi(x_3)$ ;*
- *a function $\mathbf{V}(\varphi, \psi, \theta)$ which is adjusted as in (4.86) and (4.87).*

## 4.6 Illustrative examples.

The purpose here is to illustrate the various situations which can occur through corresponding examples. In practice, we select functions $\mathfrak{L}$ and $\mathfrak{K}$ resulting from the different cases classified in Section 4.4. In each case, we produce the corresponding phases $\varphi(x)$, and also the ingredients $u$ and $v$ allowing to recover the profile $w(x, \theta)$ through (4.17) and (4.25).

To facilitate the presentation, we recall below the equations to deal with. Once $\mathfrak{L}$ and $\mathfrak{K}$ are fixed, the expression $R$ is given by (4.51) and (4.52). By construction, there exist adequate functions $\Phi$ such that

$$\partial_1 \Phi + R\, \partial_2 \Phi \equiv 0, \qquad \partial_v \Phi + R\, \partial_u \Phi \equiv 0. \tag{4.88}$$

The function $u$ must satisfy (4.26), (4.27) and (4.34), that is :

$$\left\{ \begin{array}{l} \partial_1 u + \partial_v \mathfrak{L}(u, v)\, \partial_3 u \equiv 0, \\ \partial_2 u + \partial_u \mathfrak{L}(u, v)\, \partial_3 u \equiv 0, \\ \partial_v u(x, v) = R\big(x_1, x_2, u(x, v), v\big). \end{array} \right. \tag{4.89}$$

The function $v(x, \theta)$ is obtained through (4.28), that is

$$\begin{aligned} &\partial_1 v + \partial_v \mathfrak{L}\big(u(x, v), v\big)\, \partial_3 v \\ &\qquad + \partial_v u(x, v)\, \big[\partial_2 v + \partial_u \mathfrak{L}\big(u(x, v), v\big)\, \partial_3 v\big] \equiv 0. \end{aligned} \tag{4.90}$$

Then, it becomes possible to determine $\varphi$ through (4.33). By construction, the function $\varphi$ does not depend on $\theta$ and it satisfies (4.29).

### 4.6.1 Example in the case i.1 of Lemma 4.3.

By assumption, the function $\mathfrak{L}$ is linear, say $\mathfrak{L}(u,v) = a\ u + b\ v + c$ with $(a,b,c) \in \mathbb{R}^3$. The function $R \equiv -\partial_v \mathfrak{K}/\partial_u \mathfrak{K}$ must be as indicated at the level of (4.62). To simplify, just take $R \equiv 1$ so that $\Phi = \Phi_{00}(x_2 - x_1, u - v)$. From (4.89), we deduce that $u(x,v) = \chi(x_3 - a\ x_2 - b\ x_1) + v$. On the other hand, the function $v$ can be written

$$v(x,\theta) \,=\, v_0\big(x_1 - x_2, x_3 - (a+b)\ x_2, \theta\big)\,, \qquad \partial_\theta v_0 \neq 0\,.$$

It remains to compute $\varphi(x) = \Phi_{00}\big(x_2 - x_1, \chi(x_3 - a\ x_2 - b\ x_1)\big)$.

### 4.6.2 Example in the case i.2 of Lemma 4.3.

To simplify the discussion, we work with the choice $\mathfrak{H}(t) = t$ implying that both $\mathfrak{L}$ and $\mathfrak{K}$ are functions of $u+v$. For instance, we have $\mathfrak{L}(u,v) = \mathrm{L}(u+v)$ for some function L satisfying $\mathrm{L}^{(2)} \not\equiv 0$. On the other hand, we find $R \equiv -1$ and $\Phi = \Phi_{00}(x_1 + x_2, u + v)$. Looking at (4.89), we can infer that $u(x,v)$ can be put in the form $\tilde{u}(x_1 + x_2, x_3) - v$ where $\tilde{u}(z, x_3)$ is obtained by solving the conservation law

$$\partial_z \tilde{u}(z,x_3) + \mathrm{L}'\big(\tilde{u}(z,x_3)\big)\ \partial_3 \tilde{u}(z,x_3) \,=\, 0\,, \qquad \tilde{u}(0,x_3) = \chi(x_3)\,.$$

From (4.90), we deduce that $v(x,\theta) = \tilde{v}(x_1 + x_2, x_3, \theta)$. Observe also that

$$\varphi(x) \,=\, \Phi_{00}\big(x_1 + x_2, \tilde{u}(x_1 + x_2, x_3)\big)\,.$$

### 4.6.3 Example in the case of Lemma 4.4.

The function $\mathfrak{L}$ is here linear, say $\mathfrak{L}(u,v) = a\ u + b\ v + c$ with $(a,b,c) \in \mathbb{R}^3$. The function $\mathfrak{K}$ must satisfy (4.64). We choose $\mathfrak{K}(u,v) = -\frac{1}{2}\ v^2 + u$ in order to deal with $R \equiv v$. From (4.88), we can extract that $\Phi = \Phi_{00}(2\,u - v^2)$. As expected, we see that $\Phi$ depends this time on only one variable. Moreover

$$u(x,v) \,=\, 2^{-1}\ v^2 + \chi(x_3 - a\ x_2 - b\ x_1)\,, \qquad \chi^{(1)} \neq 0\,.$$

From (4.90), we obtain that

$$v(x,\theta) \,=\, \tilde{v}(b\,x_1 - x_3, a\,x_2 - x_3, \theta)\,, \qquad \tilde{v}(y,z) \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})$$

where $\tilde{v}(y,z)$ must satisfy the Burger's law $b\,\partial_z \tilde{v} + a\,\tilde{v}\,\partial_y \tilde{v} \equiv 0$. Finally :

$$\varphi(x) \,=\, \big(\Phi_{00} \circ \chi\big)(x_3 - a\ x_2 - b\ x_1)\,.$$

### 4.6.4 Example in the case of Lemma 4.5.

The context is as in Lemma 4.5 with $\mathfrak{K} = \alpha\, u + \beta\, v + \gamma$. Choose $\mathfrak{L} = 2^{-1}\, v^2$ so that $R = -\alpha^{-1}\,(\beta + x_1)$ and $\Phi \equiv \varphi = \Phi_{00}\big(x_2 + \alpha^{-1}\,\beta\, x_1 + (2\,\alpha)^{-1}\, x_1^2\big)$. Note that $u(x,v) = -\alpha^{-1}\,\big[(\beta + x_1)\, v - x_3\big] + c$ with $c \in \mathbb{R}$. On the other hand, the function $v$ is obtained through

$$\partial_1 v \,+\, v\, \partial_3 v \,-\, \alpha^{-1}\, (x_1 + \beta)\, \partial_2 v \,\equiv\, 0\,.$$

### 4.6.5 Example in the case ii.1 of Proposition 4.4.

In agreement with (4.65) and (4.66), we can select $\mathfrak{K}(u,v) = \mathfrak{L}(u,v) = v^2 + u$ so that $R = -2\,(v + x_1)$ and $\Phi \equiv \Phi_{00}(v^2 + u + x_1^2 + 2\, x_1\, v + x_2)$. Moreover

$$u(x,v) \,=\, -\,2\, v\, x_1 - x_2 + x_3 - v^2\,, \qquad \varphi(x) \,=\, \Phi_{00}(x_1^2 + x_3)\,,$$

whereas $v(x,\theta)$ is any solution of

$$\partial_1 v \,-\, 2\, (x_1 + v)\, \partial_2 v \,-\, 2\, x_1\, \partial_3 v \,=\, 0\,.$$

### 4.6.6 Example in the case ii.2 of Proposition 4.4.

Choose $\mathfrak{K}(u,v) = u\, v^{-1}$ and $\mathfrak{L}(u,v) = (2\, v)^{-1}$ so that $R = u\, v^{-1} - x_1\, v^{-2}$. We can take

$$\Phi(x_1,x_2,u,v) \,=\, \Phi_{00}\big(\tfrac{u}{v}\, x_1 - \tfrac{x_1^2}{2\, v^2} - x_2\big)\,, \qquad u(x,v) \,=\, x_3\, v + \tfrac{x_1}{2\, v} + \alpha\, v\,.$$

The function $v$ is again solution of a suitable conservation law. On the other hand, we have again to deal with a phase $\varphi$ which is some function of a quadratic expression in $x$, namely $\varphi(x) = \Phi_{00}(x_1\, x_3 + \alpha\, x_1 - x_2)$.

### 4.6.7 Example in the case ii.3 of Proposition 4.4.

In accordance with (4.71), select

$$\mathbb{F}(t) = \frac{t^2}{2}\,, \qquad \mathbb{G}(t) = \frac{1}{2\, t}\,, \qquad \delta = 1\,, \qquad \alpha = 0\,, \qquad \mathfrak{L}(u,v) = \frac{v^2+1}{2\, u}\,.$$

From (4.27), we can deduce the implicit relation

$$u(x,v) = \tilde{U}\big(v\, x_1 - u(x,v)\, x_3, x_2, v\big)\,, \qquad \tilde{U}(X, x_2, v) \in C^1(\mathbb{R}^3;\mathbb{R})\,. \tag{4.91}$$

From (4.26) together with (4.91), we can also derive

$$\tilde{U}(X, x_2, v) = \underline{U}\big(2\, X\, \tilde{U} - (v^2+1)\, x_2, v\big)\,, \qquad \underline{U}(Y,v) \in C^1(\mathbb{R}^2;\mathbb{R})\,. \tag{4.92}$$

Use (4.91) and (4.92) in order to extract respectively $\partial_v u$ and $\partial_X \tilde{U}$. Replace $x_3$ as indicated by a function of $x_1$, $x_2$, $Y$, $v$ and $\underline{U}$. By this way, we obtain a first expression for $\partial_v u$. It is compared below with the one coming directly from (4.51)-(4.52). We find :

$$R \equiv \partial_v u \;=\; \frac{\frac{\partial_v \underline{U}}{2\,\underline{U}\,\partial_Y \underline{U}} + x_1 - \frac{v}{u}\, x_2}{\frac{v}{\underline{U}} \left[ \frac{\underline{U} - 2\,Y\,\partial_Y \underline{U}}{2\,v\,\underline{U}\,\partial_Y \underline{U}} + x_1 \right] - \frac{v^2+1}{u^2}\, x_2} \;=\; \frac{1 + x_1 - \frac{v}{u}\, x_2}{\frac{v}{\underline{U}}\,[1 + x_1] - \frac{v^2+1}{u^2}\, x_2}\,.$$

It follows that $\underline{U}(Y,v) \,=\, v \pm \sqrt{Y + v^2}$. The function $u(x,v)$ can now be deduced by just imposing $u(0,0) = 0$ together with the implicit relation :

$$u(x,v) \;=\; \underline{U}\big(2\,v\,x_1\,u(x,v) - 2\,x_3\,u(x,v)^2 - (v^2+1)\,x_2, v\big)\,.$$

On the other hand, we seek $\Phi(x_1,x_2,u,v)$ in the form

$$\Phi = \underline{\Phi}\big(u\,(1+x_1) - v\,x_2, x_2, u, v\big)\,, \qquad \underline{\Phi}(Y,x_2,u,v) \in \mathcal{C}^1(\mathbb{R}^4;\mathbb{R})\,.$$

Taking into account the preceding definition of $R$, the condition (4.36) gives rise to the equation $-\,x_2\,\partial_Y \underline{\Phi} + Y\,\partial_2 \underline{\Phi} = 0$. Thus, there is some function $\underline{\Phi}_0(X,u,v) \in \mathcal{C}^1(\mathbb{R}^3;\mathbb{R})$ such that $\underline{\Phi}(Y,x_2,u,v) = \underline{\Phi}_0(Y^2 + x_2^2, u, v)$. From (4.37), we can then deduce that $\partial_v \underline{\Phi}_0 \equiv 0$ and $2\,X\,\partial_X \underline{\Phi}_0 + u\,\partial_u \underline{\Phi}_0 = 0$. In conclusion, the following choice is suitable :

$$\Phi(x_1,x_2,u,v) \;=\; \Phi_{00}\Big(\frac{[u\,(1+x_1) -\, v\,x_2]^2 + x_2^2}{u^2}\Big)\,, \qquad \Phi_{00} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})\,.$$

With $u(x,v)$ and $\Phi(x_1,x_2,u,v)$ as above, we can deduce $\varphi(x)$ through (4.33).

# 5 The time evolution problem.

Let $(\varphi, w)$ be a compatible couple. We recall that the profile $w(x,\theta)$ can be put in the form (2.13) with a triplet $(\varphi, \psi, \mathbf{W})$ which is adjusted as it is indicated in (2.17)-(2.18)-(2.19)-(2.20) and which satisfies (2.15). The purpose of this last chapter is to explain what happens as time evolves.

## 5.1 Propagation of compatible datas.

The purpose of this paragraph 5.1 is to show the Theorem 2. Consider the system (1.11). Standard results (see for instance [15]) guarantee the existence locally in time, say on the domain $\Omega_r^T \times \mathbb{T}$ with $T \in \mathbb{R}_+^*$, of a $\mathcal{C}^1-$solution to (1.11). Introduce $\mathbf{U}(t,x,\theta) := \mathbf{W}\big(\Phi(t,x,\theta), \Psi(t,x,\theta), \theta\big)$. From (1.11), we can easily deduce that

$$(5.1) \qquad \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} = 0\,, \qquad \mathbf{U}(0,x,\theta) = \mathbf{W}\big(\varphi(x), \psi(x,\theta), \theta\big) = w(x,\theta)\,.$$

By integrating (1.11) along the associated characteristics (which are straight lines), we can exhibit the identities

$$\Phi(t,x,\theta) \;=\; \varphi\big(x - t\,\mathbf{U}(t,x,\theta)\big) \quad , \qquad \forall\,(t,x,\theta)\in\Omega_r^T\times\mathbb{T}\,, \tag{5.2}$$

$$\Psi(t,x,\theta) \;=\; \psi\big(x - t\,\mathbf{U}(t,x,\theta),\theta\big), \qquad \forall\,(t,x,\theta)\in\Omega_r^T\times\mathbb{T}\,. \tag{5.3}$$

**Lemma 5.1.** *Assume that the three ingredients $\varphi$, $\psi$ and $\mathbf{W}$ are adjusted according to (2.17)-(2.18)-(2.19)-(2.20). Then, the function $\Phi(t,x,\theta)$ issued from (1.11) is such that $\partial_\theta\Phi\equiv 0$. Moreover, noting*

$$y\equiv y(t,x) := x - t\,\mathbf{U}(t,x,\theta)\,, \qquad \Xi(y,\theta) := \big(\varphi(y),\psi(y,\theta),\theta\big)\in\mathbb{R}^2\times\mathbb{T}\,,$$

*the expression $\Psi(t,x,\theta)$ coming from (1.11) satisfies*

$$\begin{aligned}\partial_\theta\Psi(t,x,\theta) \;\equiv\; &\partial_\theta\psi(y,\theta)\\ &-t\;\nabla\psi(y,\theta)\cdot\big[\partial_\theta\mathbf{W}\big(\Xi(y,\theta)\big)+\partial_\theta\psi(y,\theta)\;\partial_\psi\mathbf{W}\big(\Xi(y,\theta)\big)\big]\,.\end{aligned} \tag{5.4}$$

***Proof of the Lemma 5.1.*** Use the relations (5.2) and (5.3) with the formula given for $\mathbf{U}$ to compute $\partial_\theta\Phi$ and $\partial_\theta\Psi$ according to

$$\mathcal{M}\,\begin{pmatrix}\partial_\theta\Phi(t,x,\theta)\\ \partial_\theta\Psi(t,x,\theta)\end{pmatrix}=\begin{pmatrix}-t\,\nabla\varphi(y)\cdot\partial_\theta\mathbf{W}\big(\Xi(y,\theta)\big)\\ \partial_\theta\psi(y,\theta)-t\;\nabla\psi(y,\theta)\cdot\partial_\theta\mathbf{W}\big(\Xi(y,\theta)\big)\end{pmatrix}$$

with a matrix $\mathcal{M}$ given by

$$\mathcal{M}(t,y,\theta)\;:=\;\begin{pmatrix}1+t\;\nabla\varphi(y)\cdot\partial_\varphi\mathbf{W} & t\;\nabla\varphi(y)\cdot\partial_\psi\mathbf{W}\\ t\;\nabla\psi(y,\theta)\cdot\partial_\varphi\mathbf{W} & 1+t\;\nabla\psi(y,\theta)\cdot\partial_\psi\mathbf{W}\end{pmatrix}.$$

In the preceding formula for the matrix $\mathcal{M}$, the functions $\partial_\star\mathbf{W}$ are evaluated at the point $\Xi(y,\theta)$. A consequence of (2.19) and (2.20) is the information: *det* $\mathcal{M}(t,y,\theta)=1$. It follows that

$$\begin{aligned}\partial_\theta\Phi(t,x,\theta)\;=\;&-t\;\nabla\varphi\cdot(\partial_\theta\mathbf{W}+\partial_\theta\psi\;\partial_\psi\mathbf{W})\\ &+t^2\;\big[(\nabla\psi\cdot\partial_\theta\mathbf{W})\,(\nabla\varphi\cdot\partial_\psi\mathbf{W})-(\nabla\varphi\cdot\partial_\theta\mathbf{W})\,(\nabla\psi\cdot\partial_\psi\mathbf{W})\big]\,.\end{aligned}$$

Note that the right hand term can be regarded as a function of $(y,\theta)$. The condition (2.17) is the same as

$$\nabla\varphi(y)\cdot\big[\partial_\theta\psi(y,\theta)\,\partial_\psi\mathbf{W}\big(\Xi(y,\theta)\big)+\partial_\theta\mathbf{W}\big(\Xi(y,\theta)\big)\big]\;=\;0\,. \tag{5.5}$$

Therefore, it remains

$$\partial_\theta\Phi(t,x,\theta)\;=\;t^2\;(\nabla\varphi\cdot\partial_\psi\mathbf{W})\;(\nabla\psi\cdot\partial_\theta\mathbf{W}+\partial_\theta\psi\;\nabla\psi\cdot\partial_\psi\mathbf{W})\,.$$

Due to (2.18), this is $\partial_\theta\Phi\equiv 0$. Proceeding as above, we can obtain

$$\begin{aligned}\partial_\theta\Psi(t,x,\theta)=\;&\partial_\theta\psi\;+\;t\;\big(\partial_\theta\psi\;\nabla\varphi\cdot\partial_\varphi\mathbf{W}-\nabla\psi\cdot\partial_\theta\mathbf{W}\big)\\ &+t^2\;\big[(\nabla\varphi\cdot\partial_\theta\mathbf{W})\,(\nabla\psi\cdot\partial_\varphi\mathbf{W})-(\nabla\varphi\cdot\partial_\varphi\mathbf{W})\,(\nabla\psi\cdot\partial_\theta\mathbf{W})\big]\,.\end{aligned}$$

Exploiting again (2.19), (2.20) and (5.5), this is equivalent to

$$\begin{aligned}\partial_\theta \Psi(t,x,\theta) = \partial_\theta \psi \ & - t \, \big( \nabla \psi \cdot \partial_\theta \mathbf{W} + \partial_\theta \psi \ \nabla \psi \cdot \partial_\psi \mathbf{W} \big) \\ & - t^2 \, (\nabla \varphi \cdot \partial_\varphi \mathbf{W}) \ \nabla \psi \cdot \big( \partial_\theta \mathbf{W} + \partial_\theta \psi \ \partial_\psi \mathbf{W} \big) \, .\end{aligned}$$

Due to (2.18) and (2.19), the term in factor of $t^2$ is necessarily equal to zero. By this way, we can see how (5.4) appears. $\square$

Consider the expression $u^\varepsilon$ which is defined on the domain $\Omega_r^T$ through

$$\begin{aligned} u^\varepsilon(t,x) &:= \mathbf{U}\Big(t, x, \tfrac{\Phi(t,x)}{\varepsilon}\Big) \\ &= \mathbf{W}\Big(\Phi(t,x), \Psi\big(t,x,\tfrac{\Phi(t,x)}{\varepsilon}\big), \tfrac{\Phi(t,x)}{\varepsilon}\Big) , \qquad \varepsilon \in \, ]0,1] \, . \end{aligned} \tag{5.6}$$

By construction, we have $u^\varepsilon(0,\cdot) \equiv h^\varepsilon(\cdot)$ with $h^\varepsilon$ given by (1.2). A direct computation based on (1.11) indicates that $u^\varepsilon(t,x)$ is indeed a solution of (1.1) on $\Omega_r^T$. By applying the Theorem 2.6 of [4], we obtain that $\big(D_x u^\varepsilon(t,x)\big)^3 \equiv 0$ on $B(0, r - t\,V)$ for all $t \in [0,T]$. Repeating at the time $t \in \, ]0,T]$ the procedure of the Section 2, we can deduce that the constraints (2.17), (2.18), (2.19) and (2.20) are propagated. In other words :

**Lemma 5.2.** *For all $t \in [0,T]$, the solutions $\Phi(t,x)$ and $\Psi(t,x,\theta)$ of (1.11) satisfy (1.13), (1.14), (1.15) and (1.16).*

These identities can also be derived by using (2.17)-(2.18)-(2.19)-(2.20) as well as (5.2), (5.3) and the Lemma 5.1. The Theorem 2 is proved.

For the sake of completeness, we can also remark that the rank of the solution is a preserved quantity. In the case of rank 1, this is obvious. In the case of rank 2, this is a consequence of what follows.

**Lemma 5.3.** *The solutions $\Phi(t,x)$ and $\Psi(t,x,\theta)$ of (1.11) satisfy*

$$\big(\nabla \Phi \wedge \nabla \Psi\big)(t,x,\theta) = \big(\nabla \varphi \wedge \nabla \psi\big)(y,\theta) \, , \qquad \forall \, (t,x,\theta) \in \Omega_r^T \times \mathbb{T} \, . \tag{5.7}$$

Thus, the volume measure $\nabla \Phi \wedge \nabla \Psi$ is constant along the characteristics. This is in fact a by-product of the divergence free relation.

***Proof of the Lemma 5.3.*** By differentiating (5.2) and (5.3) with respect to $x_j$, we can extract

$$M(t,y,\theta) \begin{pmatrix} \partial_j \Phi(t,x,\theta) \\ \partial_j \Psi(t,x,\theta) \end{pmatrix} = \begin{pmatrix} \partial_j \varphi(y) \\ \partial_j \psi(y) \end{pmatrix} , \qquad \forall \, j \in \{1,2,3\} \, .$$

It follows that

$$\begin{aligned}
(5.8) \qquad \nabla\Phi &= (1 + t\,\nabla\psi\cdot\partial_\psi\mathbf{W})\,\nabla\varphi - t\,(\nabla\varphi\cdot\partial_\psi\mathbf{W})\,\nabla\psi\,,\\
(5.9) \qquad \nabla\Psi &= -\,t\,(\nabla\psi\cdot\partial_\varphi\mathbf{W})\,\nabla\varphi + (1 + t\,\nabla\varphi\cdot\partial_\varphi\mathbf{W})\,\nabla\psi\,.
\end{aligned}$$

Now, we can use (5.8) and (5.9) in order to compute the cross product of $\nabla\Phi$ and $\nabla\Psi$. Due to (2.19) and (2.20), it remains (5.7). $\square$

## 5.2 Asymptotic phenomena.

Families like $\{u^\varepsilon\}_\varepsilon$ give many informations on the complex phenomena which may occur at the level of (1.7) when passing to the limit (as $\varepsilon \to 0$).

Noting $\mathcal{S}(t)$ with $t \in \mathbb{R}_+^*$ the semi-group operator which is associated with incompressible Euler equations, we can for instance use $\{u^\varepsilon\}_\varepsilon$ to study the well-posedness (or not) of $\mathcal{S}(t)$ in functional spaces (thus arising the delicate problem of the localization of the solutions, see [2]). We can also investigate the weak $L^2-$continuity (or not) of $\mathcal{S}(t)$ (in the spirit of [3, 12]). These applications of our current approach will not be developed in these pages. Nevertheless, we will point out some related very specific aspect.

We want here to show that the phenomenon of *superposition of oscillations* already noted in [3] (only when $d = 2$ and in the absence of the divergence free constraint) can indeed occur at the level of (1.7) when $d = 3$.

The idea is to start at the initial time $t = 0$ with a function $\psi(x)$ which does not see the variable $\theta \in \mathbb{T}$ and with a function $\mathbf{W}_\varepsilon(\cdot)$ which depends on the parameter $\varepsilon \in ]0, 1]$ and contains oscillations in the variable $\psi$, as it is indicated in (1.22). Then, in order to prove the mechanism (1.23), it suffices to exhibit some $t \in ]0, T]$ such that $\partial_\theta\Psi \neq 0$.

In view of (5.1) and because $\partial_\theta\psi \equiv 0$, it suffices to test if

$$(5.10) \qquad \exists\,(x,\theta) \in \Omega_r^0 \times \mathbb{T}\,; \qquad \partial_\theta\mathbf{W}(\varphi,\psi,\theta)\cdot\nabla\psi = \partial_\theta w\cdot\nabla\psi \neq 0\,.$$

In the framework $\nabla\varphi\cdot\partial_\psi\mathbf{W} \not\equiv 0$ of the Section 4.1, because of (4.2), it is not possible to obtain (5.10). When $\nabla\varphi\cdot\partial_\psi\mathbf{W} \equiv 0$, in the case $f' \not\equiv 0$ and $g' \not\equiv 0$, we can see with (3.15), (3.23) and (3.24) that

$$\partial_\theta w\cdot\nabla\psi \;=\; \left\{\partial_\theta\alpha\begin{pmatrix}0\\-g\\1\end{pmatrix} + \partial_\theta\beta\begin{pmatrix}1\\-f\\0\end{pmatrix}\right\}\cdot\left\{\Psi_0'\begin{pmatrix}f'(\varphi)\\0\\g'(\varphi)\end{pmatrix} + \partial_\varphi\Psi\begin{pmatrix}\partial_1\varphi\\\partial_2\varphi\\\partial_3\varphi\end{pmatrix}\right\}.$$

Taking into account (3.12) and (3.18), we must necessarily have $\partial_\theta w\cdot\nabla\psi \equiv 0$. It remains to examine the situation of the paragraph 3.2.1. The context is the one of Proposition 3.1. Choose $(a,b) \in (\mathbb{R}^*)^2$, $c=0$ and $\varphi_{00} \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$. Define $\varphi(x)$ as in (3.19). Select some function $\alpha_\varepsilon$ which is such that

$$\alpha_\varepsilon(\varphi,\psi,\theta) = A(\varphi,\psi,\psi/\varepsilon,\theta)\,, \qquad \partial_\psi A \neq 0\,, \qquad A \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathbb{T}^2;\mathbb{R})\,.$$

Choose two auxiliary functions $\phi(\theta) \in \mathcal{C}^\infty(\mathbb{T};\mathbb{R})$ and $\Psi_0(T,Z) \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$ satisfying $\phi' \neq 0$ and $\partial_T \psi_0 \neq 0$. Take

$$\chi \equiv 1\,, \quad \beta_\varepsilon(\varphi,\psi,\theta) = \alpha_\varepsilon(\varphi,\psi,\theta) - \phi(\theta)\,, \quad \Psi(X,Y,Z) = \Psi_0(Y-X,Z)\,.$$

Obviously, we have (3.22) and (3.23). With $w(x,\theta)$ defined according to (3.24), compute

$$\begin{aligned}\partial_\theta w \cdot \nabla\psi \;&=\; \left\{ \partial_\theta \alpha \begin{pmatrix} 0 \\ -b \\ 1 \end{pmatrix} + \partial_\theta \beta \begin{pmatrix} 1 \\ -a \\ 0 \end{pmatrix} \right\} \cdot \left\{ \partial_T \Psi_0 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \partial_Z \Psi_0 \begin{pmatrix} a \\ 1 \\ b \end{pmatrix} \right\} \\ &=\; \partial_T \Psi_0 \; \phi' \neq 0\,.\end{aligned}$$

In fact, the corresponging solution of (1.7) can be produced explicitly. It is

$$u^\varepsilon(t,x) \;=\; A_\varepsilon(t,x) \begin{pmatrix} 1 \\ -a-b \\ 1 \end{pmatrix} \;-\; \phi\Big(\frac{\varphi(x)}{\varepsilon}\Big) \begin{pmatrix} 1 \\ -a \\ 0 \end{pmatrix}$$

with

$$A_\varepsilon(t,x) \;:=\; A\Big(\varphi(x), \frac{\psi\big(x_3 - x_1 - t\,\phi\big(\frac{\varphi(x)}{\varepsilon}\big)\big)}{\varepsilon}, \frac{\varphi(x)}{\varepsilon}\Big)\,.$$

# 6 Appendix.

This appendix is concerned with the three-dimensional criterion which is studied at the level of Section 4.4.2. The matter is to consider the more complicated case, when $XR \not\equiv 0$ and $YR \not\equiv 0$. The Proposition 4.4 gives sufficient conditions on $\mathfrak{K}$ and $\mathfrak{L}$ in order to solve the system (4.39)-(4.40). The aim of this Appendix is to explain (under suitable assumptions that will be precised later) why there are no other possible choices.

Thus, in all this Section 6, we deal with (4.39)-(4.40) or (4.47)-(4.48), in the case $XR \not\equiv 0$ and $YR \not\equiv 0$. The starting point of our analysis is the equation (4.74). In a first approach, we assume that $\partial^2_{vv}\mathfrak{L} \not\equiv 0$. We will see in the paragraph 6.5 that the case $\partial^2_{vv}\mathfrak{L} \equiv 0$ can be dealt separately and that it does not produce other cases than (4.65).

### 6.1 Preliminary informations.

In what follows, we assume that $\partial^2_{vv}\mathfrak{L} \not\equiv 0$. In doing so, as seen below, no information is forgotten.

**Lemma 6.1.** *The assumption $\partial^2_{vv}\mathfrak{L} \equiv 0$ is not compatible with solving the system (4.39)-(4.40) in the case $(XR)\,(YR) \not\equiv 0$.*

***Proof of Lemma 6.1.*** First, consider the case when $\partial^2_{vv}\mathfrak{L} \equiv 0$ and also $\partial^2_{uv}\mathfrak{L} \not\equiv 0$. To this end, introduce $c(u,v) := \partial^2_{uu}\mathfrak{L}/\partial^2_{uv}\mathfrak{L}$. The notations are as in (4.73). The starting point of the discussion is the identity (4.74) which can here be reduced to

$$\frac{d_v\mathfrak{K}}{\partial^2_{uv}\mathfrak{L}} + R_{\tilde v}\; x_1 + \big[c(U,V)\,R_{\tilde v} + 1\big]\; x_2 - Q_{\tilde v}\; v\; \big[c(U,V)\,R^2_{\tilde v} + 2\;R_{\tilde v}\big] \equiv 0\,.$$

Since $\partial^2_{vv}\mathfrak{L} \equiv 0$ and $\partial^2_{uv}\mathfrak{L} \not\equiv 0$, we are sure that $\partial_1 R_{\tilde v} \not\equiv 0$. Thus, we can work with $R_{\tilde v},\ x_2,\ u,\ v, \tilde v$ instead of $x_1,\ x_2,\ u,\ v, \tilde v$. In particular, for $R_{\tilde v} = 0$, it remains

$$\partial_v\mathfrak{K}(u,V)\;\partial^2_{uv}\mathfrak{L}(u,V)^{-1} + x_2 \;\equiv\; 0\,, \qquad \forall\,(x_2,u,V)\,, \qquad V = v + \tilde v\,.$$

This is not possible since the three variables $x_2$, $u$ and $V$ are independent. Thus, we have necessarily $\partial^2_{uv}\mathfrak{L} \equiv 0$. Knowing that $\partial^2_{vv}\mathfrak{L} \equiv \partial^2_{uv}\mathfrak{L} \equiv 0$ and that $XR \not\equiv 0$, the function $\mathfrak{L}$ can be put in the form $F(u) + b\,v$ for some constant $b \in \mathbb{R}$ and some function $F \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ satisfying $F^{(2)} \not\equiv 0$. This time, the identity (4.74) becomes

$$d_v\mathfrak{K}(U,V) + x_2\;F^{(2)}(U)\;R_{\tilde v} - v\;Q_{\tilde v}\;F^{(2)}(U)\;R^2_{\tilde v} \;\equiv\; 0\,. \tag{6.1}$$

In particular, for $R_{\tilde v} = 0$, we find $\partial_v\mathfrak{K} \equiv 0$, that is $\mathfrak{K}(u,v) \equiv \tilde{\mathfrak{K}}(u)$. Then, dividing (6.1) by $R_{\tilde v}$ and taking $v = 0$, we obtain $\tilde{\mathfrak{K}}'(u) + x_2\,F^{(2)}(u) \equiv 0$. Since $F^{(2)} \neq 0$, this furnishes the expected contradiction. □

From now on, assume that $\partial^2_{vv}\mathfrak{L} \not\equiv 0$. Introduce the two auxiliary functions

$$a(u,v) \;:=\; \partial^2_{uv}\mathfrak{L}/\partial^2_{vv}\mathfrak{L}\,, \qquad b(u,v) \;:=\; \partial^2_{uu}\mathfrak{L}/\partial^2_{vv}\mathfrak{L}\,. \tag{6.2}$$

From the informations (4.51)-(4.52) written with $v = \tilde v$, we obtain that

$$\big[a(u,\tilde v)\,R_{\tilde v} + 1\big]\;\partial_2 R_{\tilde v} \;-\; \big[b(u,\tilde v)\,R_{\tilde v} + a(u,\tilde v)\big]\;\partial_1 R_{\tilde v} \;=\; 0\,. \tag{6.3}$$

Now, the idea is to manipulate (4.74) in order to eliminate the contribution

$$d_v\mathfrak{K} \;\equiv\; R_{\tilde v}\;\partial_u\mathfrak{K}(u + R_{\tilde v}\,v, \tilde v + v) \;+\; \partial_v\mathfrak{K}(u + R_{\tilde v}\,v, \tilde v + v)\,.$$

To this end, it suffices to apply the vector field $\partial_2 R_{\tilde v}\,\partial_1 - \partial_1 R_{\tilde v}\,\partial_2$ to the equation (4.74). Then, use (6.3) in order to extract

$$
\begin{aligned}
\Xi(u,v,\tilde v,R_{\tilde v}) \;:=\; & \big[a(u,\tilde v) - a(U,V)\big] + \big[b(u,\tilde v) - b(U,V)\big]\; R_{\tilde v} \\
& + \big[a(U,V)\,b(u,\tilde v) - a(u,\tilde v)\,b(U,V)\big]\; R_{\tilde v}^2 \\
& - v\;\tilde\chi(x_1,x_2,u,\tilde v)\big[1 + 2\,a(U,V)\,R_{\tilde v} + b(U,V)\,R_{\tilde v}^2\big] \;=\; 0
\end{aligned}
\tag{6.4}
$$

with

$$\tilde\chi(x_1,x_2,u,\tilde v) \;:=\; \big[b(u,\tilde v)\,R_{\tilde v} + a(u,\tilde v)\big]\;\partial_1 Q_{\tilde v} \;-\; \big[a(u,\tilde v)\,R_{\tilde v} + 1\big]\;\partial_2 Q_{\tilde v}\,.$$

By definition, the function $\tilde\chi$ does not depend on $v$. On the other hand, in view of (6.4), it is an expression of the four variables $u$, $v$, $\tilde v$ and $R_{\tilde v}$ (which are independent because $XR \not\equiv 0$). It means that

$$\exists\,\chi \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R})\,; \qquad \tilde\chi(x_1,x_2,u,\tilde v) \;=\; \chi(u,\tilde v,R_{\tilde v})\,. \tag{6.5}$$

**Lemma 6.2.** *There exist functions $f$, $g$, $k$ and $l$ in $\mathcal{C}^2(\mathbb{R};\mathbb{R})$ such that*

$$a(u,v) = f(u)\,v + g(u)\,, \qquad b(u,v) = (2\,f^2 - f')(u)\,v^2 + k(u)\,v + l(u) \tag{6.6}$$

*where the expression $\mathbf{Z}(u) := 2\,f(u)^2 - f'(u)$ must satisfy*

$$\mathbf{Z}'(u) \;=\; 2\;f(u)\;\mathbf{Z}(u)\,. \tag{6.7}$$

***Proof of Lemma 6.2.*** As indicated line (6.5), we can replace $\tilde\chi$ by $\chi$ at the level of (6.4). Then, compute

$$\partial^2_{vv}\Xi(u,v,\tilde v,0) \;=\; -\,\partial^2_{vv}a(u,\tilde v + v) \;=\; 0\,.$$

Thus, we can find functions $f$ and $g$ such that $a(u,v) = f(u)\,v + g(u)$. On the other hand $\partial_v\Xi(u,v,\tilde v,0) \equiv -\partial_v a(u,\tilde v + v) - \chi(u,\tilde v,0) \equiv 0$ which implies that $\chi(u,\tilde v,0) = -f(u)$. It follows that

$$
\begin{aligned}
0 \;=\; \partial_{R_{\tilde v}}\Xi(u,v,\tilde v,0) \;=\; & b(u,\tilde v) - b(u,\tilde v + v) \\
& + v\;(\tilde v + v)\;\big[2\,f(u)^2 - f'(u)\big] \\
& + v\;\big[2\,f(u)\,g(u) - g'(u)\big] - v\;\partial_{R_{\tilde v}}\chi(u,\tilde v,0)\,.
\end{aligned}
\tag{6.8}
$$

In particular, we must have

$$\partial^3_{R_{\tilde v}vv}\Xi(u,v,\tilde v,0) \;=\; -\partial^2_{vv}b(u,\tilde v + v) + 4\;f(u)^2 - 2\;f'(u) \;=\; 0\,.$$

In other words, there are two functions $k$ and $l$ such that the second part of (6.6) is verified. Coming back to (6.8), we can see that

$$\partial_{R_{\tilde v}}\chi(u,\tilde v,0) \;=\; -\,\mathbf{Z}(u)\;\tilde v \;-\; k(u) \;+\; 2\;f(u)\;g(u) \;-\; g'(u)\,.$$

Then look at the condition

$$
\begin{aligned}
0 \;=\; (\partial_{R_{\tilde v}})^2\, \Xi(u,v,\tilde v,0) \;=\; & -v^2\ \partial^2_{uu} a(u,V) - 2\ v\ \partial_u b(u,V) \\
& + 2\ \big[ a(u,V)\, b(u,\tilde v) - a(u,\tilde v)\, b(u,V) \big] \,-\, v\ (\partial_{R_{\tilde v}})^2\, \chi(u,\tilde v,0) \\
& - 4\, v\, \partial_{R_{\tilde v}} \chi(u,\tilde v,0)\, a(u,V) + 2\, v\, f(u)\, \big[ 2\, v\, \partial_u a(u,V) + b(u,V) \big]\,.
\end{aligned} \tag{6.9}
$$

Taking into account the preceding informations on $a$ and $b$, the expression $(\partial_{R_{\tilde v}})^2\, \Xi(u,v,\tilde v,0)$ is a polynomial function with respect to $v$. In particular, the coefficient in factor of $v^3$ must be zero. This criterion yields (6.7). $\square$

From (6.4), we can extract a formula for $\chi$, namely

$$
\chi(u,\tilde v,R_{\tilde v}) \;= \tilde{\mathfrak{P}}_1(u,\tilde v,v,R_{\tilde v})\ \tilde{\mathfrak{P}}_2(u,\tilde v,v,R_{\tilde v})^{-1}
$$

where $\tilde{\mathfrak{P}}_1$ and $\tilde{\mathfrak{P}}_2$ are polynomial functions in $R_{\tilde v}$. Work in a neighbourhood of $\mathbb{R}^2$ where $v \neq 0$. Since $\chi$ does not depend on $v$, we must have

$$
\big[ (v^3\, \partial_v \tilde{\mathfrak{P}}_1)\ (v\, \tilde{\mathfrak{P}}_2) - (v^2\, \tilde{\mathfrak{P}}_1)\ (v^2\, \partial_v \tilde{\mathfrak{P}}_2) \big]\ v^{-4}\ \tilde{\mathfrak{P}}_2^{-4} \;\equiv\; 0\,. \tag{6.10}
$$

Replace $R_{\tilde v}$ by $(U-u)/v$. Then, change the point of view by adopting $u$, $U$, $\tilde v$ and $v$ as being the new (independant) variables. Noting simply $\tilde a$ and $a^*$ when the function $a$ is evaluated at the points $(u,\tilde v)$ and $(U,\tilde v)$, the condition (6.10) becomes

$$
\mathfrak{D}(u,\tilde v,U,v) \;:=\; (\mathfrak{P}_1\, \mathfrak{P}_2 - \mathfrak{P}_3\, \mathfrak{P}_4)(u,\tilde v,U,v) \;=\; 0\,.
$$

More precisely

$$
\begin{aligned}
\mathfrak{P}_1 \;=\; & v^3\ [-f(U) - (U-u)\ \mathbf{Z}(U)] \\
+\; & v^2\ [\tilde a - a^* - 2\ (U-u)\ \mathbf{Z}(U)\ \tilde v - (U-u)\ k(U) - (U-u)^2\ \tilde a\ \mathbf{Z}(U)] + \\
+\; & v\ \Big[ (U-u)\ (\tilde b - b^*) + (U-u)^2\ \big( \tilde b\ f(U) - 2\ \tilde a\ \mathbf{Z}(U)\ \tilde v - \tilde a\ k(U) \big) \Big] + \\
+\; & \big[ (U-u)^2\ (\tilde b\ a^* - \tilde a\ b^*) \big]\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_2 \;=\; & v^2\ [1 + 4\ (U-u)\ f(U) + 2\ (U-u)^2\ f^{(1)}(U) + 3\ (U-u)^2\ \mathbf{Z}(U) \\
& \qquad + (U-u)^3\ \mathbf{Z}^{(1)}(U)] \\
+\; & v\ \Big[ (U-u)^3\ \big( k^{(1)}(U) + 2\ \mathbf{Z}^{(1)}(U)\ \tilde v \big) \\
& \qquad + (U-u)^2\ \big( 2\ \partial_u a^* + 4\ \mathbf{Z}(U)\ \tilde v + 2\ k(U) \big) + 2\ (U-u)\ a^* \Big] \\
+\; & \big[ \partial_u b^* (U-u)^3 + b^* (U-u)^2 \big]\,,
\end{aligned}
$$

$$\begin{aligned}
\mathfrak{P}_3 \;&=\; v^2\,[1+2\,(U-u)\,f(U)+(U-u)^2\,\mathbf{Z}(U)] \\
&+\; v\,\big[2\,(U-u)\,a^*+2\,(U-u)^2\,\mathbf{Z}(U)\,\tilde{v}+(U-u)^2\,k(U)\big] \\
&+\; [(U-u)^2\,b^*]\,,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_4 \;&=\; v^3\,\Big[-f(U)-(U-u)\,\big(f^{(1)}(U)+2\,\mathbf{Z}(U)\big)-(U-u)^2\,\mathbf{Z}^{(1)}(U)\Big] \\
&+\; v^2\,\Big[(U-u)\,\big(-\partial_u a^*-2\,\mathbf{Z}(U)\,\tilde{v}-k(U)\big) \\
&\qquad +(U-u)^2\,\big(-2\,\mathbf{Z}^{(1)}(U)\,\tilde{v}-k^{(1)}(U)-2\,\tilde{a}\,\mathbf{Z}(U)\big) \\
&\qquad +(U-u)^3\,\big(-\tilde{a}\,\mathbf{Z}^{(1)}(U)\big)\Big] \\
&+\; v\,\Big[(U-u)^2\,\big(-\partial_u b^*+\tilde{b}\,f(U)-2\,\tilde{a}\,\mathbf{Z}(U)\,\tilde{v}-\tilde{a}\,k(U)\big) \\
&\qquad +(U-u)^3\,\big(-\tilde{a}\,k^{(1)}(U)+\tilde{b}\,f^{(1)}(U)-2\,\tilde{a}\,\mathbf{Z}^{(1)}(U)\,\tilde{v}\big)\Big] \\
&+\; [\tilde{b}\,\partial_u a^*-\tilde{a}\,\partial_u b^*]\,(U-u)^3\,.
\end{aligned}$$

All expressions $\mathfrak{P}_j$ are polynomial functions in $v$ and $\tilde{v}$, with degree at most 5 in $v$ and 3 in $\tilde{v}$. It follows that $\mathfrak{D}$ can be put in the form

$$\mathfrak{D}(u,\tilde{v},U,v)=\sum_{j=0}^{5}\,\mathfrak{D}_j(u,\tilde{v},U)\,v^j\equiv 0,\quad \mathfrak{D}_j(u,\tilde{v},U)=\sum_{k=0}^{3}\mathfrak{D}_j^k(u,U)\,\tilde{v}^k\,.$$

Of course, the condition $\mathfrak{D}\equiv 0$ amounts to the same thing as

$$\mathfrak{D}_j^k\;\equiv\;0\,,\qquad \forall\,(j,k)\in\{0,\cdots,5\}\times\{0,\cdots,3\}\,. \tag{6.11}$$

**Lemma 6.3.** *Concerning the structures of the functions* $\mathbf{Z}$ *and* $f$*, there are three possible situations:*
*a)* $\mathbf{Z}\equiv 0$ *and* $f\equiv 0$*;*
*b) There exists a constant* $\delta\in\mathbb{R}$ *such that* $\mathbf{Z}\equiv 0$ *and* $f(u)=(\delta-2\,u)^{-1}$*;*
*c) There exist two constants* $(\delta_1,\delta_2)\in\mathbb{R}^2$ *such that* $\mathbf{Z}(u)=(u^2-2\,\delta_1\,u+\delta_2)^{-1}$ *and* $f(u)=(-u+\delta_1)\,(u^2-2\,\delta_1\,u+\delta_2)^{-1}$.

***Proof of Lemma 6.3.*** When $\mathbf{Z}\equiv 0$, the function $f$ must satisfy $2\,f^2-f'\equiv 0$ and we are faced with situations *a)* or *b)*.

From now on, suppose that $\mathbf{Z}\not\equiv 0$. First, look at the coefficient $\mathfrak{D}_0^5$ which is

$$\begin{aligned}
\mathfrak{D}_0^5 \;=\; (U-u)^4\times\Big\{&\big[\mathbf{Z}(u)\,f(U)-f(u)\,\mathbf{Z}(U)\big]\,\mathbf{Z}(U) \\
&+(U-u)\,\mathbf{Z}(u)\,\big[\mathbf{Z}^{(1)}(U)\,f(U)-\mathbf{Z}(U)\,f^{(1)}(U)\big]\Big\}\,.
\end{aligned}$$

Since $\mathbf{Z} \not\equiv 0$, dividing the expression $\mathfrak{D}_0^5$ by $\mathbf{Z}(u)\ \mathbf{Z}(U)^2$, the condition $\mathfrak{D}_0^5 \equiv 0$ becomes

$$\frac{f(U)}{\mathbf{Z}(U)} - \frac{f(u)}{\mathbf{Z}(u)} - (U-u)\left[\frac{f(U)}{\mathbf{Z}(U)}\right]^{(1)} = 0\,.$$

It means that the function $f\,\mathbf{Z}^{-1}$ is linear with respect to $U$. Combining this information with (6.7), it first remains

$$\frac{f(U)}{\mathbf{Z}(U)} = -U + \delta_1, \qquad \delta_1 \in \mathbb{R} \tag{6.12}$$

and then, replacing $f$ by $\mathbf{Z}'/(2\,\mathbf{Z})$, we have access to *c)*. □

## 6.2 Discussion of the case $\mathbf{Z} \not\equiv 0$.

This paragraph 6.2 is aimed to be the source of the situation **ii.3** described in Proposition 4.4. First, by changing $u$ into $u-\delta_1$ and defining $\gamma := \delta_2 - \delta_1^2$, we can always assume that

$$f(u) \;=\; -\,u/(u^2+\gamma)^{-1}\,, \qquad \mathbf{Z}(u) \;=\; (u^2+\gamma)^{-1}\,. \tag{6.13}$$

The functions $a$ and $b$ are given by (6.6). We have to determine $g$, $k$ and $l$.

**Lemma 6.4.** *Assume that $Z \not\equiv 0$. Then, there are constants $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$ and $\delta \in \mathbf{R}$ such that*

$$g(u) \;=\; -\,\frac{\alpha\, u - \beta}{u^2+\gamma}\,, \qquad k(u) \;=\; \frac{2\,\alpha}{u^2+\gamma}\,, \qquad l(u) \;=\; \frac{\delta}{u^2+\gamma}\,. \tag{6.14}$$

***Proof of Lemma 6.4.*** Complete (6.2) with the introduction of

$$\tilde{c}(u,v) \;:=\; \partial_v \mathfrak{K}/\partial^2_{vv}\mathfrak{L}\,, \qquad \tilde{d}(u,v) \;:=\; \partial_u \mathfrak{K}/\partial^2_{vv}\mathfrak{L}\,. \tag{6.15}$$

We have here to work with

$$R \;=\; -\,\frac{R_1}{R_2} \;:=\; -\,\frac{\tilde{c}(u,v) + x_1 + a(u,v)\ x_2}{\tilde{d}(u,v) + a(u,v)\ x_1 + b(u,v)\ x_2}\,. \tag{6.16}$$

It is easy to check that $1 + a\,R \not\equiv 0$. Otherwise $R = a(u,v)^{-1}$ so that $XR \equiv 0$ which is in contradiction with the assumptions of Proposition 4.4. From (6.16), we can extract

$$x_1 \;=\; -\Big\{R\ \tilde{d}(u,v) + \tilde{c}(u,v) + x_2\ \big[R\ b(u,v) + a(u,v)\big]\Big\}/\big[R\ a(u,v)+1\big]\,.$$

Use the relation (6.16) to compute $Q(x_1,x_2,u,v) \equiv YR/XR$. Then replace $x_1$ as it is indicated above in order to obtain

$$Q(x_1,x_2,u,v) \;=\; \mathfrak{Q}(R,x_2,u,v) \;=\; \frac{\mathfrak{Q}_0(R,u,v)}{\mathfrak{Q}_2(R,u,v)} \,+\, \frac{\mathfrak{Q}_1(R,u,v)}{\mathfrak{Q}_2(R,u,v)}\; x_2$$

with

$$\mathfrak{Q}_i(R,u,v) \;=\; \sum_{j=0}^{3} \mathfrak{Q}_i^j(u,v)\; R^j\,, \qquad i\in\{0,1,2\}$$

and

$$\begin{array}{ll}
\mathfrak{Q}_1^3 = \partial_u b\; a - \partial_u a\; b\,, & \mathfrak{Q}_1^2 = a\;\partial_v b + \partial_u b - \partial_v a\; b\,,\\
\mathfrak{Q}_1^1 = \partial_v b + \partial_u a\,, & \mathfrak{Q}_1^0 = \partial_v a\,,\\
\mathfrak{Q}_0^3 = \partial_u \tilde d\; a - \partial_u a\;\tilde d\,, & \mathfrak{Q}_0^2 = \partial_u \tilde d + a^2\;\partial_v(\tfrac{d}{a}) + a^2\;\partial_u(\tfrac{c}{a})\,,\\
\mathfrak{Q}_0^1 = a\;\partial_v \tilde c + \partial_v \tilde d + \partial_u \tilde c - \partial_v a\;\tilde c\,, & \mathfrak{Q}_0^0 = \partial_v \tilde c\,.\\
\mathfrak{Q}_2^3 = a\; b\,, & \mathfrak{Q}_2^2 = b + 2\,a^2\,,\\
\mathfrak{Q}_2^1 = 3\;a\,, & \mathfrak{Q}_2^0 = 1\,.
\end{array}$$

The condition (4.48) is the same as $\partial_u R - Q\,\partial_2 R + XQ \equiv 0$. Compute

$$\begin{aligned}
\partial_u R \;&=\; -\,\frac{p_u}{p_0}\; := \;-\;\frac{\partial_u \tilde c + \partial_u a\; x_2 + R\;[\partial_u \tilde d + \partial_u a\; x_1 + \partial_u b\; x_2]}{\tilde d + a\; x_1 + b\; x_2}\,,\\
\partial_2 R \;&=\; -\,\frac{p_2}{p_0}\; := \;-\;\frac{a + b\; R}{\tilde d + a\; x_1 + b\; x_2}\,,\\
XR \;&=\; -\,\frac{p_x}{p_0}\; := \;-\;\frac{1 + 2\; a\; R + b\; R^2}{\tilde d + a\; x_1 + b\; x_2}\,.
\end{aligned}$$

With these conventions, the condition (4.48) becomes

$$-\,p_u + \mathfrak{Q}\; p_2 - \partial_R \mathfrak{Q}\; p_x - R_1\;\partial_2 \mathfrak{Q} \;\equiv\; 0\,. \tag{6.17}$$

Observe that (6.17) is linear with respect to the variable $x_2$. In particular, the term in factor of $x_2$ must be zero implying that

$$\begin{aligned} &-\,\mathfrak{Q}_2\;(\partial_u a + \partial_u b\; R) \;-\; \partial_R \mathfrak{Q}_1\;(R^2\; b + 2\; a\; R + 1)\\ &\qquad +\partial_u a\; R\;(R\; b + a)\;(R^2\; b + 2\; a\; R + 1) + 4\;\mathfrak{Q}_1\;(R\; b + a) \equiv 0\,. \end{aligned} \tag{6.18}$$

Since $\partial_1 R \;\not\equiv 0$ (because $1 + a\,R \not\equiv 0$), the four variables $R$, $x_2$, $u$ and $v$ are independent. The left hand side of (6.18) is a polynomial function in $R$ of degree 4 whose four coefficients must all be zero.

Applying this criterion, we can derive the two following conditions :

$$2\; b\;\partial_u a - 2\; a\;\partial_u b + b\;\partial_v b \equiv 0\,, \qquad -\partial_u b + 2\; b\;\partial_v a \equiv 0\,. \tag{6.19}$$

From (6.6), (6.13) and (6.19), we can derive the expressions of $k$ and $l$ given in (6.14). Then, it suffices to exploit (6.19) to obtain $g$.

$\square$

At this stage, the functions $a$ and $b$ are determined. From the first part of (6.2), we can deduce the existence of functions $\mathbb{F}$, $\mathbb{G}$ and $\mathfrak{g}$ such that

$$\mathfrak{L}(u,v) \,=\, (u^2+\gamma)^{1/2}\; \mathbb{F}\big((v+\alpha)\;(u^2+\gamma)^{-\,1/2}+\mathfrak{g}(u)\big) \,+\, \mathbb{G}(u) \tag{6.20}$$

where $\mathbb{F}^{(2)} \not\equiv 0$ and $\mathfrak{g}$ is some primitive of the function $u \longmapsto \beta\,(u^2+\gamma)^{-\,3/2}$. Then, testing the second part of (6.2) with the formula (6.20), we can see that necessarily $\beta = \gamma = 0$, yielding the form (4.70) for $\mathfrak{L}$. As already observed at the level of the proof of Proposition 4.4, once $\mathfrak{L}$ is given by (4.70) with $\mathbb{F}$ and $\mathbb{G}$ as in (4.71), the choice $\mathfrak{K} = \partial_v \mathfrak{L}$ is suitable. The existence of other convenient functions $\mathfrak{K}$ will not be discussed.

## 6.3 Exclusion of the case $\mathbf{Z} \equiv 0$ and $f \not\equiv 0$.

In this paragraph, we consider the situation *b)* of Lemma 6.3. When $\mathbf{Z} \equiv 0$, we can compute the quantities $\mathfrak{D}^k_j$ of (6.11) to find the following list :

$$\begin{aligned}
\mathfrak{D}^0_5 \,:=\,& 2\,(U-u)\,[f^{(1)}(U)-2\,f^2(U)]\,, \\
\mathfrak{D}^1_4 \,:=\,& f(u)-f(U) \,+\, (U-u)\,[4\,f(U)\,f(u)-4\,f^2(U)+f^{(1)}(U)] \\
&+(U-u)^2\,[2\,f^{(1)}(U)\,f(u)]\,, \\
\mathfrak{D}^0_4 \,:=\,& g(u)-g(U) \,+\, (U-u)\,\big[-4\,f(U)\,g(U)+g^{(1)}(U)+4\,f(U)\,g(u)\big] \\
&+(U-u)^2\,\big[2\,f^{(1)}(U)\,g(u)+k^{(1)}(U)-3\,f(U)\,k(U)\big] \\
&+(U-u)^3\,\big[k^{(1)}(U)\,f(U)-k(U)\,f^{(1)}(U)\big]\,, \\
\mathfrak{D}^2_3 \,:=\,& 2\,(U-u)\,[f(u)\,f(U)-f^2(U)+(U-u)\,f^{(1)}(U)\,f(u)]\,, \\
\mathfrak{D}^1_3 \,:=\,& (U-u)\,\big[2\,f(u)\,g(U)-4\,g(U)\,f(U)+2\,f(U)\,g(u)+k(u)-k(U)\big] \\
&+(U-u)^2\,\big[2\,g^{(1)}(U)\,f(u)-6\,f(U)\,k(U)+2\,k(U)\,f(u) \\
&\qquad\qquad +2\,g(u)\,f^{(1)}(U)+4\,k(u)\,f(U)+k^{(1)}(U)\big] \\
&+(U-u)^3\,\big[2\,k^{(1)}(U)\,f(u)+2\,k^{(1)}(U)\,f(U)-2\,f^{(1)}(U)\,k(U) \\
&\qquad\qquad +2\,f^2(U)\,k(u)-2\,f(u)\,f(U)\,k(U)+f^{(1)}(U)\,k(u)\big] \\
&+(U-u)^4\,\big[2\,f(U)\,f(u)\,k^{(1)}(U)-2\,f^{(1)}(U)\,f(u)\,k(U)\big]\,, \\
\mathfrak{D}^2_2 \,:=\,& (U-u)^2\,[3\,k(u)\,f(U)-3\,k(U)\,f(U)] \\
&+(U-u)^3\,[2\,k^{(1)}(U)\,f(u)+k^{(1)}(U)\,f(U)+4\,k(u)\,f^2(U) \\
&\qquad\qquad -4\,k(U)\,f(U)\,f(u)+k(u)\,f^{(1)}(U)-k(U)\,f^{(1)}(U)] \\
&+(U-u)^4\,[4\,f(U)\,f(u)\,k^{(1)}(U)-4\,f(u)\,f^{(1)}(U)\,k(U)]\,,
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{D}_3^0 &:= (U-u)\,\big[2\,g(u)\,g(U) - 2\,g^2(U) + l(u) - l(U)\big]\\
&\quad + (U-u)^2\,\big[2\,g^{(1)}(U)\,g(u) + 2\,k(U)\,g(u) - 2\,k(U)\,g(U)\\
&\qquad\qquad +4\,f(U)\,l(u) - 4\,f(U)\,l(U) + l^{(1)}(U)\big]\\
&\quad + (U-u)^3\,\big[l^{(1)}(U)\,f(U) + 2\,k^{(1)}(U)\,g(u) + k^{(1)}(U)\,g(U)\\
&\qquad\qquad -g^{(1)}(U)\,k(U) - f^{(1)}(U)\,l(U) - 2\,f(U)\,k(U)\,g(u)\\
&\qquad\qquad +2\,l(u)\,f^2(U) + f^{(1)}(U)\,l(u) - k^2(U)\big]\\
&\quad + (U-u)^4\,\big[2\,g(u)\,f(U)\,k^{(1)}(U) - 2\,f^{(1)}(U)\,g(u)\,k(U)\big]\,,\\
\mathfrak{D}_2^1 &:= 3\,(U-u)^2\,[-f(U)\,l(U) - k(U)\,g(U) + k(u)\,g(U) + l(u)\,f(U)]\\
&\quad + (U-u)^3\,[2\,l^{(1)}(U)\,f(u) + f(U)\,l^{(1)}(U) - 2\,k^2(U) + 2\,k^{(1)}(U)\,g(u)\\
&\qquad\qquad +k^{(1)}(U)\,g(U) + k(u)\,g^{(1)}(U) + 2\,k(u)\,k(U) - k(U)\,g^{(1)}(U)\\
&\qquad\qquad +4\,g(U)\,k(u)\,f(U) + 4\,l(u)\,f^2(U) - 4\,f(U)\,k(U)\,g(u)\\
&\qquad\qquad +l(u)\,f^{(1)}(U) - l(U)\,f^{(1)}(U) - 4\,f(U)\,f(u)\,l(U)]\\
&\quad + (U-u)^4\,[k(u)\,k^{(1)}(U) + k(U)\,k(u)\,f(U) - 2\,g^{(1)}(U)\,k(U)\,f(u)\\
&\qquad\qquad -f(u)\,k^2(U) - 4\,f^{(1)}(U)\,g(u)\,k(U) - 2\,f^{(1)}(U)\,f(u)\,l(U)\\
&\qquad\qquad +2\,f(U)\,f(u)\,l^{(1)}(U) + 4\,f(U)\,g(u)\,k^{(1)}(U) + 2\,g(U)\,f(u)\,k^{(1)}(U)]\\
&\quad + (U-u)^5\,[k(u)\,f(U)\,k^{(1)}(U) - k(U)\,k(u)\,f^{(1)}(U)]\,,\\
\mathfrak{D}_2^0 &:= (U-u)^2\,[3\,l(u)\,g(U) - 3\,l(U)\,g(U)]\\
&\quad + (U-u)^3\,[2\,l^{(1)}(U)\,g(u) + l^{(1)}(U)\,g(U) - 2\,k(U)\,l(U) + l(u)\,g^{(1)}(U)\\
&\qquad\qquad +2\,k(U)\,l(u) - l(U)\,g^{(1)}(U) + 4\,f(U)\,g(U)\,l(u) - 4\,g(u)\,l(U)\,f(U)]\\
&\quad + (U-u)^4\,[l(u)\,k^{(1)}(U) + l(u)\,f(U)\,k(U) - 2\,g^{(1)}(U)\,g(u)\,k(U) - k^2(U)\,g(u)\\
&\qquad\qquad -2\,f^{(1)}(U)\,l(U)\,g(u) + 2\,f(U)\,l^{(1)}(U)\,g(u) + 2\,g(U)\,g(u)\,k^{(1)}(U)]\\
&\quad + (U-u)^5\,[l(u)\,f(U)\,k^{(1)}(U) - l(u)\,k(U)\,f^{(1)}(U)]\,,\\
\mathfrak{D}_1^3 &:= (U-u)^3\,[2\,k(u)\,f^2(U) - 2\,f(u)\,f(U)\,k(U)]\\
&\quad + (U-u)^4\,[2\,f(u)\,f(U)\,k^{(1)}(U) - 2\,f(u)\,k(U)\,f^{(1)}(U)]\,,\\
\mathfrak{D}_1^2 &:= (U-u)^3\,[k(u)\,k(U) - k^2(U) + 4\,k(u)\,f(U)\,g(U) - 2\,f(u)\,g(U)\,k(U)\\
&\quad +2\,l(u)\,f^2(U) - 2\,g(u)\,f(U)\,k(U) - 2\,f(u)\,f(U)\,l(U)]\\
&\quad + (U-u)^4\,[k(u)\,k^{(1)}(U) + 3\,k(u)\,k(U)\,f(U) - 2\,f(u)\,k^2(U)\\
&\qquad\qquad -2\,f(u)\,k(U)\,g^{(1)}(U) - 2\,g(u)\,k(U)\,f^{(1)}(U)\\
&\qquad\qquad -2\,f(u)\,l(U)\,f^{(1)}(U) + 2\,f(u)\,g(U)\,k^{(1)}(U)\\
&\qquad\qquad +2\,f(u)\,f(U)\,l^{(1)}(U) + 2\,g(u)\,f(U)\,k^{(1)}(U) - k(u)\,f(U)\,k(U)]\\
&\quad + (U-u)^5\,[2\,k(u)\,f(U)\,k^{(1)}(U) - 2\,k(u)\,k(U)\,f^{(1)}(U)]\,,\\
\mathfrak{D}_1^0 &:= (U-u)^3\,[l(u)\,l(U) - l^2(U) + 2\,l(u)\,g^2(U) - 2\,g(u)\,g(U)\,l(U)]\\
&\quad + (U-u)^4\,[l(u)\,l^{(1)}(U) - 2\,g(u)\,k(U)\,l(U) + 2\,l(u)\,g(U)\,k(U)\\
&\qquad\qquad -2\,g(u)\,l(U)\,g^{(1)}(U) + 2\,g(u)\,g(U)\,l^{(1)}(U)]\\
&\quad + (U-u)^5\,[l(u)\,f(U)\,l^{(1)}(U) + l(u)\,g(U)\,k^{(1)}(U)\\
&\qquad\qquad -l(u)\,k(U)\,g^{(1)}(U) - l(u)\,l(U)\,f^{(1)}(U)]\,,\\
\mathfrak{D}_0^3 &:= (U-u)^4\,[k(u)\,k(U)\,f(U) - f(u)\,k^2(U)]\\
&\quad + (U-u)^5\,[k(u)\,k^{(1)}(U)\,f(U) - k(u)\,k(U)\,f^{(1)}(U)]\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_1^1 := {}& (U-u)^3\,[k(u)\,l(U) - 2\,k(U)\,l(U) + l(u)\,k(U) + 2\,k(u)\,g^2(U) \\
&\qquad + 4\,f(U)\,g(U)\,l(u) - 2\,f(u)\,l(U)\,g(U) \\
&\qquad - 2\,f(U)\,g(u)\,l(U) - 2\,g(u)\,g(U)\,k(U)] \\
&+ (U-u)^4\,[k(u)\,l^{(1)}(U) - 2\,f(u)\,k(U)\,l(U) + l(u)\,k^{(1)}(U) + 2\,l(u)\,k(U)\,f(U) \\
&\qquad - 2\,k^2(U)\,g(u) + 2\,k(u)\,k(U)\,g(U) - 2\,f(u)\,l(U)\,g^{(1)}(U) \\
&\qquad - 2\,g^{(1)}(U)\,g(u)\,k(U) - 2\,g(u)\,l(U)\,f^{(1)}(U) + 2\,f(U)\,l^{(1)}(U)\,g(u) \\
&\qquad + 2\,g(U)\,f(u)\,l^{(1)}(U) + 2\,g(U)\,g(u)\,k^{(1)}(U)] \\
&+ (U-u)^5\,[k(u)\,f(U)\,l^{(1)}(U) + k^{(1)}(U)\,k(u)\,g(U) - k(U)\,k(u)\,g^{(1)}(U) \\
&\qquad -2\,k(U)\,l(u)\,f^{(1)}(U) - l(U)\,k(u)\,f^{(1)}(U) + 2\,l(u)\,f(U)\,k^{(1)}(U)]\,, \\
\mathfrak{D}_0^2 := {}& (U-u)^4\,[k(u)\,f(U)\,l(U) - 2\,f(u)\,k(U)\,l(U) + k(u)\,k(U)\,g(U) \\
&\qquad + l(u)\,k(U)\,f(U) - g(u)\,k^2(U)] \\
&+ (U-u)^5\,[k(u)\,f(U)\,l^{(1)}(U) + k(u)\,k^{(1)}(U)\,g(U) + l(u)\,f(U)\,k^{(1)}(U) \\
&\qquad - k(u)\,k(U) g^{(1)}(U) - l(u)\,k(U)\,f^{(1)}(U) - k(u)\,l(U)\,f^{(1)}(U)]\,, \\
\mathfrak{D}_0^1 := {}& (U-u)^4\,[k(u)\,g(U)\,l(U) + l(u)\,f(U)\,l(U) - f(u)\,l^2(U) \\
&\qquad - 2\,g(u)\,k(U)\,l(U) + l(u)\,g(U)\,k(U)] \\
&+ (U-u)^5\,[k(u)\,l^{(1)}(U)\,g(U) + l(u)\,l^{(1)}(U)\,f(U) + l(u)\,g(U)\,k^{(1)}(U) \\
&\qquad - l(u)\,l(U)\,f^{(1)}(U) - k(u)\,l(U)\,g^{(1)}(U) - l(u)\,k(U)\,g^{(1)}(U)]\,, \\
\mathfrak{D}_0^0 := {}& (U-u)^4\,[l(u)\,g(U)\,l(U) - g(u)\,l^2(U)] \\
&+ (U-u)^5\,[l(u)\,l^{(1)}(U)\,g(U) - l(u)\,l(U)\,g^{(1)}(U)]\,.
\end{aligned}
$$

We start by obtaining preliminary informations on $g$, $k$ and $l$.

**Lemma 6.5.** *In the case $\mathbf{Z} \equiv 0$ and $f(u) = (\delta - 2\,u)^{-1}$, we can find a constant $\beta \in \mathbb{R}$ and a function $\mathfrak{Q} \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ such that*

$$
g(u) \;=\; \mathfrak{Q}(u)\,f(u)^2\,, \qquad \mathfrak{Q}^{(2)}(u) = 2\,\beta, \tag{6.21}
$$

$$
k(u) \;=\; \beta\,f(u)\,, \tag{6.22}
$$

$$
l(u) \;=\; \beta\,g(u)\,. \tag{6.23}
$$

***Proof of Lemma 6.5.*** Since $f \not\equiv 0$, the condition $\mathfrak{D}_1^3 \equiv 0$ amounts to the same thing as

$$
\frac{k(u)}{f(u)} \;-\; \frac{k(U)}{f(U)} \;+\; (U-u)\,\Big(\frac{k(U)}{f(U)}\Big)^{(1)} \;\equiv\; 0\,.
$$

In other words $k(u) = (\alpha\,u + \beta)\,f(u)$ with $(\alpha,\beta) \in \mathbb{R}^2$. On the other hand, the restriction $\mathfrak{D}_0^3 \equiv 0$ is equivalent to

$$
\frac{k(u)}{f(u)}\;\frac{k(U)}{f(U)} \;-\; \frac{k(U)^2}{f(U)^2} \;+\; (U-u)\;\frac{k(u)}{f(u)}\;\Big(\frac{k(U)}{f(U)}\Big)^{(1)} \;=\; 0\,.
$$

This is possible only if $\alpha = 0$, yielding (6.22).

Knowing (6.22), the constraint $\mathfrak{D}_4^0 \equiv 0$ allows to extract

$$\frac{g(u)}{f(u)^2} \;=\; \Big[\frac{g(U)}{f(U)^2} + 4\,U\,\frac{g(U)}{f(U)} - U\,\frac{g^{(1)}(U)}{f(U)^2} + U^2\,\beta\Big] + \Big[-4\,\frac{g(U)}{f(U)} + \frac{g^{(1)}(U)}{f(U)^2} - 2\,U\,\beta\Big]\,u + \beta\,u^2\,.$$

Since the left hand side of this identity does not depend on the variable $U$, the coefficients of the right hand side must be constants. We recognize here what is said at the level (6.21). In view of of the informations (6.21) and (6.22), the condition $\mathfrak{D}_1^2 \equiv 0$ becomes

$$\begin{aligned}2\,l(u)\,f(U)^2 \;=\; & -\beta^2\,f(u)\,f(U) + \beta^2\,f(U)^2 \\ & -2\,\beta\,\mathfrak{Q}(U)\,f(u)\,f(U)^3 + 2\,\beta\,\mathfrak{Q}(u)\,f(u)^2\,f(U)^2 \\ & +2\,f(u)\,f(U)\,l(U) + (U-u)\,\big\{-2\,\beta^2\,f(u)\,f(U)^2 \\ & \quad +2\,\beta\,\mathfrak{Q}^{(1)}(U)\,f(u)\,f(U)^3 + 4\,\beta\,\mathfrak{Q}(U)\,f(u)\,f(U)^4 \\ & \quad +4\,l(U)\,f(u)\,f(U)^2 - 2\,f(u)\,f(U)\,l^{(1)}(U)\big\}\,.\end{aligned} \tag{6.24}$$

Dividing this identity by $2\,f(u)^2\,f(U)^2$, the left hand side is simply $l(u)\,f(u)^{-2}$ whereas the right hand side is some polynomial function in $u$ of degree 2. To better visualize the content of (6.24), we can work with the auxiliary variables $X := \delta - 2\,u$ and $Y := \delta - 2\,U$. Observe that

$$\mathfrak{Q}(u) \;=\; \check{\mathfrak{Q}}(X) \;=\; \frac{\beta}{4}\,X^2 + \check{\mathfrak{Q}}_1\,X + \check{\mathfrak{Q}}_0\,,$$

$$l(u)\,f(u)^{-2} \;=\; \widetilde{\mathfrak{Q}}(X) \;=\; \frac{\widetilde{\mathfrak{Q}}_2}{4}\,X^2 + \widetilde{\mathfrak{Q}}_1\,X + \widetilde{\mathfrak{Q}}_0\,.$$

The relation (6.24) is the same as

$$\begin{aligned}2\,\widetilde{\mathfrak{Q}}(X)\,Y^2 \;=\; & -\beta^2\,X\,Y^3 + \beta^2\,X^2\,Y^2 - 2\,\beta\,\check{\mathfrak{Q}}(Y)\,X\,Y + 2\,\beta\,\check{\mathfrak{Q}}(X)\,Y^2 \\ & +2\,\widetilde{\mathfrak{Q}}(Y)\,X\,Y + (X-Y)\,\big[-\beta^2\,X\,Y^2 + \beta\,(\check{\mathfrak{Q}})^{(1)}(Y)\,X\,Y \\ & +2\,\beta\,\check{\mathfrak{Q}}(Y)\,X - (\widetilde{\mathfrak{Q}})^{(1)}(Y)\,X\,Y - 2\,\widetilde{\mathfrak{Q}}(Y)\,X\big]\,.\end{aligned}$$

For $X = 0$ and $Y \neq 0$, we obtain $\widetilde{\mathfrak{Q}}_0 = \beta\,\check{\mathfrak{Q}}_0$. Examining the coefficients in factor of $X^2\,Y^2$ and $X\,Y^2$, we get respectively $\widetilde{\mathfrak{Q}}_2 = \beta^2$ and $\widetilde{\mathfrak{Q}}_1 = \beta\,\check{\mathfrak{Q}}_1$. In other words we have $\widetilde{\mathfrak{Q}} = \beta\,\check{\mathfrak{Q}}$, that is (6.23). □

The above study does not exploit all the informations which are contained in the coefficients $\mathfrak{D}_*^\star$. We can go further

**Lemma 6.6.** *In the case* $\mathbf{Z} \equiv 0$ *and* $f(u) = (\delta - 2\,u)^{-1}$, *we must have* $k \equiv 0$ *and* $l \equiv 0$. *On the other hand, we must have* $g(u) = d\,f(u)$ *with* $d \in \mathbb{R}$.

***Proof of Lemma 6.6.*** Knowing (6.22) and (6.23), the coefficient $\mathfrak{D}_1^0$ can be simplified into $\beta^2 \left[g(u)\ g(U) - g^2(U) + (U-u)\ g(u)\ g^{(1)}(U)\right] = 0$. If $\beta \neq 0$, exploiting (6.21), this is possible only if

$$\begin{aligned} &2\ \check{\mathfrak{Q}}(X)\ \check{\mathfrak{Q}}(Y)\ Y^2 \ -\ 2\ \check{\mathfrak{Q}}(Y)^2\ X^2 \\ &\qquad\qquad + (X-Y)\ \check{\mathfrak{Q}}(X)\ \left[\check{\mathfrak{Q}}^{(1)}(Y)\ Y^2 + 4\ \check{\mathfrak{Q}}(Y)\ Y\right] \ =\ 0\,. \end{aligned}$$

For $X \neq 0$ and $Y = 0$, we obtain $\check{\mathfrak{Q}}_0 = 0$. Then, dividing by $Y^2$ and taking again $X \neq 0$ and $Y = 0$, we can see that $\check{\mathfrak{Q}}_1 = 0$ and also $\beta = 0$. It means that $k \equiv 0$ and $l \equiv 0$. Then, look at the condition $\mathfrak{D}_3^0 \equiv 0$ which is

$$g(u)\,g(U) - g(U)^2 + (U-u)\,g'(U)\,g(u) \ =\ 0\,.$$

Necessarily, we must have $g(u) = d\,(e-2\,u)^{-1}$ for some $(d,e) \in \mathbb{R}^2$. Taking into account (6.21), the only possible choice for the constant $e$ is $e = \delta$ giving rise to the expected result. □

**Lemma 6.7.** *The case $\mathbf{Z} \equiv 0$ and $f(u) = (\delta - 2\,u)^{-1}$ must be excluded.*

***Proof of Lemma 6.7.*** If $F' \equiv 0$, we have $G^{(2)} \equiv 0$ and $\mathfrak{L}$ is linear in $u$ and $v$. This is in contradiction with the assumptions $X R \not\equiv 0$ and $Y R \not\equiv 0$. Therefore $F' \not\equiv 0$, $F'(v) = c\,(v+d)^{-2}$ and $G^{(2)}(v) = \delta\,c\,(v+d)^{-3}$ for some $c \in \mathbb{R}^*$. Now, consider the relation (4.74) which can be simplified into

$$\begin{aligned} d_v\mathfrak{K} + x_1\ &\left[F^{(1)}(V)\ R_{\tilde v} + F^{(2)}(V)\ U + G^{(2)}(V)\right] + x_2\ F^{(1)}(V) \\ &- v\ Q_{\tilde v}\ \left[2\ R_{\tilde v}\ F^{(1)}(V) + F^{(2)}(V)\ U + G^{(2)}(V)\right] = 0\,. \end{aligned} \tag{6.25}$$

Multiply this expression by $(V+d)^3$ and then take $V = -d$. It remains

$$c\ (\delta - 2\,U)\ \left(x_1 + (d+\tilde v)\ Q_{\tilde v}\right) \ =\ 0\,, \qquad c \in \mathbb{R}^*\,.$$

It means that $Q_{\tilde v} = -x_1\,(d+\tilde v)^{-1}$. Replace $Q_{\tilde v}$ accordingly at the level of (6.25). Multiply the expression thus obtained by $(d+\tilde v)\,F'(V)^{-1}$ and then take $\tilde v = -d$ in order to obtain $2\,R_{\tilde v}\,v - 2\,U + \delta = -2\,u + \delta = 0$ which cannot be satisfied for all $u$. This is again a contradiction showing finally that the case $\mathbf{Z} \equiv 0$ and $f \not\equiv 0$ must indeed be excluded. □

## 6.4 Discussion of the case $f \equiv 0$.

When $f \equiv 0$, the structure of the coefficients $\mathfrak{D}_*^\star$ is simplified. It becomes easier to understand what contains the system (6.11).

**Lemma 6.8.** *In the case $f \equiv 0$, the functions $k$, $l$, and $g$ must satisfy one of the two distinct following restrictions:*

*a) We have $k \equiv 0$, $g^{(1)} \equiv 0$ and $l^{(1)} \equiv 0$;*

*b) There exists a constant $\Theta \in \mathbb{R}^*$ such that $k \equiv -2\,\Theta$, $g^{(1)} \equiv \Theta$ and $l \equiv 0$.*

***Proof of Lemma 6.8.*** We start by looking at the condition

$$\mathfrak{D}_4^0(u,U) = g(u) - g(U) + (U-u)\, g^{(1)}(U) + (U-u)^2\, k^{(1)}(U) \,=\, 0\,.$$

Apply $\partial^2_{uu}$ to see that $g^{(2)} = -2\, k^{(1)} = \tilde{\delta}$ with $\tilde{\delta} \in \mathbb{R}$. Then, consider the restriction $\mathfrak{D}_2^1 \equiv 0$ which can be rewritten

$$\begin{aligned} &-3\, k(U)\, g(U) + 3\, k(u)\, g(U) \\ &\qquad + (U-u)\, \big[-2\, k(U)^2 + 2\, k^{(1)}(U)\, g(u) + k^{(1)}(U)\, g(U) \\ &\qquad\qquad + k(u)\, g^{(1)}(U) + 2\, k(u)\, k(U) - k(U)\, g^{(1)}(U)\big] \\ &\qquad + (U-u)^2\, \big[k(u)\, k^{(1)}(U)\big] \,=\, 0\,. \end{aligned}$$

Apply $\partial^2_{uu}$ and replace $U$ by $u$ to deduce that $g^{(2)} \equiv \tilde{\delta} = 0$. The next step is to look at the relation $\mathfrak{D}_3^0 \equiv 0$ which becomes

$$\begin{aligned} &2\, g(u)\, g(U) - 2\, g(U)^2 + l(u) - l(U) \\ &\quad + (U-u)\, \big[2\, g^{(1)}(U)\, g(u) + 2\, k(U)\, g(u) - 2\, k(U)\, g(U) + l^{(1)}(U)\big] \\ &\quad + (U-u)^2\, \big[-g^{(1)}(U)\, k(U) - k(U)^2\big] \,=\, 0\,. \end{aligned}$$

Apply again $\partial^2_{uu}$ and replace $U$ by $u$. Since $\tilde{\delta} = 0$, we obtain by this way that $l$ is a polynomial function of degree at most 2, say $l(u) = l_2\, u^2 + l_1\, u + l_0$ with $(l_0, l_1, l_2) \in \mathbb{R}^3$. Now, the condition $\mathfrak{D}_0^0 \equiv 0$ says that

$$\begin{aligned} &l(u)\, g(U)\, l(U) - g(u)\, l^2(U) \\ &\quad + (U-u)\, l(u)\, \big[l^{(1)}(U)\, g(U) - l(U)\, g^{(1)}(U)\big] \,=\, 0\,. \end{aligned} \tag{6.26}$$

Composing (6.26) on the left with $\partial^3_{uuu}$ and replacing $U$ by $u$ gives rise to

$$l^{(2)}\, \big[l^{(1)}\, g - l\, g^{(1)}\big] \,\equiv\, 0\,.$$

At this stage, we claim that $l^{(2)} \equiv 2\, l_2 \equiv 0$. To see why this is true, first assume that $l_2 \neq 0$. Due to the above relation, we must have $g = c\, l$ with $c \in \mathbb{R}$. We cannot have $c \in \mathbb{R}^*$ because then $0 \equiv g^{(2)} \equiv c\, l^{(2)}$ and therefore $l^{(2)} \equiv 0$ which is a contradiction. Retain that $c = 0$ so that $g \equiv 0$. Looking at the condition $\mathfrak{D}_1^0 \equiv 0$, we get

$$l(u)\, l(U) - l^2(U) + (U-u)\, l(u)\, l^{(1)}(U) \,\equiv\, 0\,, \qquad l(u) = l_2\, u^2 + l_1\, u + l_0$$

which is not possible when $l_2 \neq 0$. We must have $l_2 \equiv 0$ and $l(u) = l_1\, u + l_0$.

Now, we can come back to (6.26). Apply $\partial^2_{uu}$ to (6.26) and replace $U$ by $u$. It remains $l^{(1)}\, \big[l^{(1)} g - l g^{(1)}\big] \equiv 0$. We claim that $l^{(1)} \equiv l_1 \equiv 0$. Indeed, if $l_1 \neq 0$, we must have $g = c\, l$ with $c \in \mathbb{R}$ and the condition $\mathfrak{D}_1^0 \equiv 0$ becomes

$$\begin{aligned} &l(u)\, l(U) - l(U)^2 + (U-u)\, l(u)\, l^{(1)}(U) \\ &\qquad - (U-u)^2\, c\, k(U)\, l(u)\, l^{(1)}(U) \,\equiv\, 0\,. \end{aligned}$$

Since $l^{(2)} \equiv 0$, applying $\partial^2_{uu}$ and replacing $U$ by $u$ gives rise to

$$l^{(1)}(u)\ \big[l^{(1)}(u) + c\,k(u)\,l(u)\big] \ \equiv\ 0\,.$$

Recalling that $k^{(1)} \equiv 0$, we can see that this is not possible when $l^{(1)} \not\equiv 0$. In other words, the functions $k$ and $l$ are constants, say $k = \mathrm{k} \in \mathbb{R}$ and $l = \mathrm{l} \in \mathbb{R}$. The condition $\mathfrak{D}^1_1 \equiv 0$ can be rewritten

$$\begin{aligned} &2\ \mathrm{k}\ g(U)^2 - 2\ \mathrm{k}\ g(u)\ g(U) \\ &\qquad + (U-u)\ \big[-2\ \mathrm{k}^2\ g(u) + 2\ \mathrm{k}^2\ g(U) - 2\ \mathrm{k}\, g^{(1)}(U)\ g(u)\big] \\ &\qquad - (U-u)^2\ \big[\mathrm{k}^2\ g^{(1)}(U)\big] \ \equiv\ 0\,. \end{aligned}$$

Apply $\partial^2_{uu}$ and replace $U$ by $u$ in order to extract

$$\mathrm{k}\ g^{(1)}(u)\ \big[\mathrm{k} + 2\ g^{(1)}(u)\big] \ \equiv\ 0\,. \tag{6.27}$$

Now look at the restriction $\mathfrak{D}^0_3 \equiv 0$ which amounts to the same thing as

$$\begin{aligned} &\big[2\ g(u)\ g(U) - 2\ g(U)^2\big] \\ &\qquad + (U-u)\ \big[2\ g^{(1)}(U)\ g(u) + 2\ \mathrm{k}\ g(u) - 2\ \mathrm{k}\ g(U)\big] \\ &\qquad - (U-u)^2\ \big[\mathrm{k}\, g^{(1)}(U) + \mathrm{k}^2\big] \ \equiv\ 0\,. \end{aligned}$$

Combining this information with (6.27), we obtain that either $g^{(1)} \equiv 0$ and $k \equiv 0$, or that $k \equiv -2\ g^{(1)} \equiv -2\,\Theta$ with $\Theta \in \mathbb{R}^*$. We recover here the two situations *a)* and *b)* described in Lemma 6.8. In the case *b)*, we can even obtain more informations about $l$. Indeed, the condition $\mathfrak{D}^0_1 \equiv 0$ says that

$$\begin{aligned} \mathrm{l}\ \Big\{ & g(U)^2 - g(u)\ g(U) + (U-u)\ \big[\mathrm{k}\, g(U) - \mathrm{k}\, g(u) - g(u)\ g^{(1)}(U)\big] \\ & + 2^{-1}\,(U-u)^2\ \big[\mathrm{k}\ g^{(1)}(U)\big] \Big\} \ \equiv\ 0\,. \end{aligned}$$

Apply $\partial^2_{uu}$ and replace $U$ by $u$ to get $\mathrm{l}\,\big[2\ g^{(1)}(u)^2 + 3\ \mathrm{k}\ g^{(1)}(u)\big] \equiv -\,4\,\mathrm{l}\,\Theta^2 \equiv 0$. It follows that $l \equiv 0$ as expected. $\square$

Solving the system (6.11) is necessary but not sufficient in order to get solutions of (4.39)-(4.40). A direct study is still required. We start by examining the case *b)* of Lemma 6.8.

**Lemma 6.9.** *The case $k \equiv -2\ \Theta$, $g^{(1)} \equiv \Theta$, $l \equiv 0$ with $\Theta \in \mathbb{R}^*$ is excluded.*

***Proof of Lemma 6.9.*** Taking into account (6.2), we have

$$\partial^2_{uv}\mathfrak{L} = (\Theta\ u + \Theta_0)\ \partial^2_{vv}\mathfrak{L}\,, \qquad \partial^2_{uu}\mathfrak{L} = -\,2\ \Theta\ v\ \partial^2_{vv}\mathfrak{L}\,, \qquad \Theta \neq 0\,, \qquad \Theta_0 \in \mathbb{R}\,.$$

To simplify the presentation, we will adopt in this paragraph the following conventions $A := \partial_v \mathfrak{K} / \partial^2_{vv} \mathfrak{L}$ and $B := \partial_u \mathfrak{K} / \partial^2_{vv} \mathfrak{L}$. We have to deal with

$$R \,=\, - \,\frac{q_0}{p_0} \,=\, - \,\frac{A + x_1 + (\Theta\ u + \Theta_0)\ x_2}{B + (\Theta\ u + \Theta_0)\ x_1 - 2\ \Theta\ v\ x_2}\,.$$

We can compute

$$\begin{aligned}
\partial_2 R \,&=\, - \,\frac{p_2}{p_0} \,=\, - \,\frac{\Theta\ u + \Theta_0 - 2\ \Theta\ v\ R}{p_0}\,, \\
\partial_u R \,&=\, - \,\frac{p_u}{p_0} \,=\, - \,\frac{\partial_u A + \Theta\ x_2 + (\partial_u B + \Theta\ x_1)\ R}{p_0}\,,
\end{aligned}$$

and similar expressions for $\partial_1 R$ and $\partial_v R$. Briefly, the quantity $Q$ is given by

$$Q \,=\, \frac{p_y}{p_x} \,=\, \frac{\partial_v A + [\partial_u A + \partial_v B - \Theta\ x_2]\ R + [\partial_u B + \Theta\ x_1]\ R^2}{1 + 2\ [\Theta\ u + \Theta_0]\ R - 2\ \Theta\ v\ R^2}\,.$$

Now, we can write $XQ = -\,p_Q\, p_x^{-1}\ p_0^{-1}$ with

$$\begin{aligned}
p_Q \,:=\, & [\partial_u A + \partial_v B - \Theta\ x_2 + 2\ (\partial_u B + \Theta\ x_1)\ R]\ p_x \\
& - [2(\Theta\ u + \Theta_0) - 4\ \Theta\ v\ R]\ p_y\,.
\end{aligned}$$

In this context, the condition (4.48) becomes $p_u\, p_x - p_y\, p_2 + p_Q \equiv 0$. Now, the idea is to multiply this quantity by $p_0^3$ and to replace everywhere $p_0 R$ by $-q_0$ in order to obtain that some polynomial function in $x_1$ and $x_2$ with coefficients depending on $u$ and $v$ must be zero. In particular, the coefficient in factor of $x_1^4$ must be zero. This criterion yields $\Theta\ \big[(\Theta\ u{+}\Theta_0)^2{+}2\,\Theta\, v\big] \equiv 0$. This is not possible due to the assumption $\Theta \in \mathbb{R}^*$. $\square$

## 6.5 Necessary conditions.

We can summarize what has been obtained above and in the preceding Sections 6.1, 6.3 and 6.4 through the following statement.

**Proposition 6.1.** *When $(XR)\,(YR) \not\equiv 0$, the system (4.39)-(4.40) can be solved if and only if $\partial^2_{vv} \mathfrak{L} \not\equiv 0$. Then, there are two functions $f$ and $g$ in $\mathcal{C}^2(\mathbb{R};\mathbb{R})$ such that $\partial^2_{uu} \mathfrak{L}(u,v) = \big[f(u)\, v{+}g(u)\big]\, \partial^2_{vv} \mathfrak{L}$. Defining $\mathbf{Z} := 2\, f^2{-}f'$, we have either $\mathbf{Z} \equiv 0$ or $\mathbf{Z}(u) = (u^2 - 2\,\delta_1\, u + \delta_2)^{-1}$ with $(\delta_1, \delta_2) \in \mathbb{R}^2$. In the case $\mathbf{Z} \equiv 0$, the function $\mathfrak{L}$ is necessarily of the form (4.65) or (4.68).*

***Proof of Proposition 6.1.*** The first part is a repetition of Lemma 6.3. When $\mathbf{Z} \equiv 0$, due to Lemmas 6.7, 6.8 and 6.9, there are two constants $b \in \mathbb{R}$ and $c \in \mathbb{R}$ such that $\partial^2_{uv} \mathfrak{L} - b\ \partial^2_{vv} \mathfrak{L} \equiv 0$ and $\partial^2_{uu} \mathfrak{L} - c\ \partial^2_{vv} \mathfrak{L} \equiv 0$.

The first relation indicates that

$$\mathfrak{L}(u,v) \, = \, \mathbb{G}(u) + \mathbb{F}(b\, u + v)\,, \qquad (\mathbb{F},\mathbb{G}) \in \mathcal{C}^2(\mathbb{R};\mathbb{R})^2\,, \qquad \mathbb{F}^{(2)} \not\equiv 0\,.$$

Plugging this expression of $\mathfrak{L}$ in the second relation, we obtain

$$\mathbb{G}^{(2)}(u) \, + \, (b^2 - c)\, \mathbb{F}^{(2)}(b\, u + v) \, \equiv \, 0\,.$$

When $b^2 - c \neq 0$, since the variables $u$ and $b\, u + v$ are independent, both functions $\mathbb{F}^{(2)}$ and $\mathbb{G}^{(2)}$ must be constants and we recover (4.65). On the contrary, when $b^2 - c = 0$, there is no restriction on $\mathbb{F}$ (apart from $\mathbb{F}^{(2)} \not\equiv 0$) but the function $\mathbb{G}$ must be linear in $u$. It follows that $\mathfrak{L}(u,v)$ can be put in the form (4.68). $\square$

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